“An Empirical Comparison among VaR Models and Time Rules with Elliptical and Stable Distributed Returns”

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AN EMPIRICAL COMPARISON AMONG VAR MODELS AND TIME RULES WITH ELLIPTICAL AND STABLE DISTRIBUTED RETURNS[1]

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Abstract

This paper compares and investigates the impact of different VaR models with conditional elliptical and stable distributed returns. In particular, we analyze some non-Gaussian VaR models and discuss the applicability of some temporal aggregation rules. Thus, we propose and examine the performance of several VaR models: (i) an EWMA model with Student's *t* conditional distributions, (ii) a stable sub-Gaussian model, (iii) a stable asymmetric model. All models are subjected to backtest on out-of-sample data in order to assess their forecasting power and to show how the associated aggregation rules are performed in practice.

Key words: Elliptical distributions, stable distributions, time aggregation rules, backtest analysis.

JEL Classification: G21, C32, C53.

Introduction

This paper proposes an empirical comparison among several models used to value the risk of a given portfolio. In particular, we test the capacity of different VaR non-Gaussian models and their associate time rules to predict future losses of some financial positions.

The Value at Risk represents the minimum loss among the worst $(1-\theta)\%$ cases that could occur in a given temporal horizon. Even if this risk measure is not an example of coherent risk (according to Artzner et al., 1999), it is actually used by financial institutions to evaluate the market risk exposure of their trading portfolios. The approaches proposed in the literature to value the risk of a given portfolio are mainly parametric or non-parametric ones. This paper presents parametric models for heavy tailed return series. These models are based on a distributional assumption of the financial returns and they generally permit on-line VaR calculation. This feature allows the non-expert investor to understand the risk associated with his/her position.

In the RiskMetrics model (see Longerstaey and Zangari, 1996), the conditional profit/loss distribution is normal, though several empirical studies show that it is not Gaussian. As a matter of fact, the kurtosis and the skewness found in many empirical analyses led researchers to reject the normal assumption of historical conditional return series. Furthermore, the asymptotic behavior of conditional financial returns (see, among others, Mandelbrot, 1963a-b; Fama, 1965; Rachev and Mittnik, 2000 and references therein) validates and justifies the assumption of conditional stable distributed returns.

Following these studies, our paper presents and compares some alternative models for the computation of VaR considering their time scale transformations and the distributional characteristics of return series (see Lamantia et al., 2006). Firstly, we focus our attention on returns either with conditional multivariate elliptical distributions or in the domain of attraction of stable laws. For each stable model, we describe a method to estimate all parameters. Then, we compare the performance of all symmetric and asymmetric VaR models proposed and their time aggregation

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rules. In particular, we evaluate Value at Risk estimates of all models considering conditional and unconditional coverage backtesting methods (see, among others, Christoffersen, 1998; Christoffersen and Diebold, 2000; Berkowitz, 2001). Thus, the main contribution of this paper consists in an assessment of several distributional assumptions and the relative time rules to compute the Value at Risk of a given portfolio.

The paper is organized as follows: in Section 2, we recall the main VaR models and the respective time rules we test. In Section 3, we backtest the proposed VaR models assessing their ability to capture extreme returns. Section 4 discusses the applicability of different time rules comparing different results with historical and simulated data. Finally, we briefly summarize the paper.

2. Value at Risk with elliptical EWMA models and stable Paretian distributions

One of the most widely used models to compute Value at Risk of a given portfolio is the RiskMetrics one (see Longerstaey and Zangari, 1996). This model assumes that the conditional distribution of the continuously compounded return is a Gaussian law. In particular, if we denote with $w = [w_1, ..., w_n]'$ the vector of the positions taken in *n* assets forming the portfolio, then the return portfolio at time *t*+1 is given by

$$z_{(p),t+1} = \sum_{i=1}^{n} w_i z_{i,t+1},$$

where $z_{i,t+1} = \log\left(P_{i,t+1} / P_{i,t}\right)$ is the return of *i*-th asset during the period [t,t+1], and $P_{i,t}$ is the price of *i*-th asset at time *t*. RiskMetrics assumes that the return vector $z_{t+1} = \left[z_{1,t+1}, ..., z_{n,t+1}\right]'$ follows a conditional joint Gaussian distribution with null mean and variance and covariance matrix $Q_{t+1/t} = \left[\sigma^2_{ij,t+1/t}\right]$, where $\sigma^2_{ij,t+1/t}$ are estimated using the exponential weighting moving average (EWMA) model:

$$\sigma^2_{ii,t+1/t} = E_t(z^2_{i,t+1}) = \lambda\sigma^2_{ii,t/t-1} + (1-\lambda)z^2_{i,t}, \quad (1)$$

$$\sigma^2_{ij,t+1/t} = E_t(z_{i,t+1}z_{j,t+1}) = \lambda\sigma^2_{ij,t/t-1} + (1-\lambda)z_{i,t}z_{j,t}, \quad (2)$$

where λ is the optimal smoothing factor (see Longerstaey and Zangari, 1996).

2.1. Elliptical EWMA model

The simplest generalization of Gaussian EWMA model (see Lamantia et al., 2006) assumes conditional joint elliptically distributed vectors of returns $z_{t+1} = \Sigma_{t+1/t}\varepsilon_{t+1} \sim Ell_n(0, Q_{t+1/t}, f)$ with finite variance covariance matrix $Q_{t+1/t} = \left[\sigma^2_{ij,t+1/t}\right] = \Sigma_{t+1/t}\Sigma'_{t+1/t}$ that evolves as in formulas (1) and (2) and conditional characteristic function $\Phi_{z_{t+1}}(m) = E_t(e^{im'z_{t+1}}) = f(m'Q_{t+1/t}m)$. Under the elliptical assumption for the conditional returns, the Value at Risk of portfolio $z_{(p),t+1}$ at (1-θ)% (denoted by $VaR_{\theta,t+1/t}$) is simply given by

$$VaR_{\theta,t+1/t}(z_{(p),t+1}) = k_{1,1-\theta}\sigma_{(p),t+1/t}. \quad (3)$$

where $k_{1,1-\theta}$ is the tabulated value of the corresponding elliptical percentile $Ell_1(0,1,f)$ and $\sigma^2_{(p),t+1/t} = w'Q_{t+1/t}w$ is the conditional variance of portfolio $z_{(p),t+1}$. In addition, we can also study temporal aggregation rules of EWMA models considering the aggregated returns $Z_{t+T} = \sum_{s=1}^{T} z_{t+s}$. The vector Z_{t+T} follows an ISR-SARV(1) process and we generally do not know its distribution (see Meddahi and Renault, 2004; Lamantia et al., 2006). However, if we require that the vectors of returns z_{t+s} are i.i.d. (independent identically distributed) elliptically $Ell_n(0, Q_{t+1/t}, f)$ distributed, then even the sum $Z_{t+T} = \sum_{s=1}^{T} z_{t+s}$ is elliptically distributed, but with a different distributional law $Z_{t+T} \sim Ell_n(0, \tilde{Q}_{t+T/t}, \tilde{f})$ uniquely determined by a different characteristic generator $\tilde{f}$. In this case, the variance-covariance matrix of Z_{t+T} at time t for the elliptical model is given by $\tilde{Q}_{t+T/t} = TQ_{t+1/t}$ where $Q_{t+1/t}$ is the covariance matrix of i.i.d. elliptically $Ell_n(0, Q_{t+1/t}, f)$ distributed n-dimensional vector of returns z_{t+s}. Thus we can apply the variance temporal rule to estimate at time t the (1-θ)% VaR in the periods [t,t+1] and [t,t+T]. Then, $VaR_{\theta,t+1/t}(z_{(p),t+1}) = k_{1,1-\theta}\sigma_{(p),t+1/t}$ and the temporal aggregation rule

$$VaR_{\theta,t+T/t} \approx k_{2,1-\theta}\sqrt{T}\sigma_{(p),t+1/t} = M_\theta \sqrt{T} VaR_{\theta,t+1/t} \tag{4}$$

hold where $M_\theta = \dfrac{k_{2,1-\theta}}{k_{1,1-\theta}}$, and $k_{1,1-\theta}$, $k_{2,1-\theta}$ are respectively the corresponding 1-θ elliptical $Ell_1(0,1,f); Ell_1(0,1,\tilde{f})$ percentiles. In particular, for T big enough $k_{2,1-\theta}$ tends to the percentile of the standard Gaussian, while if $Ell_1(0,1,f) \sim N(0,1)$ is a standard normal then also $Ell_1(0,1,\tilde{f}) \sim N(0,1)$ is Gaussian and we obtain the RiskMetrics time rule $VaR_{\theta,t+T/t} \approx \sqrt{T} VaR_{\theta,t+1/t}$.

However, this time rule is not valid for the EWMA model and we underline that the use of time rules of any strong GARCH-type process can be applied only as approximation and for limited temporal horizons T because the distributional structure of the aggregated process generally changes (see Lamantia et al., 2006).

2.2. Stable EWMA model

Alternative to the elliptical EWMA model with finite variance, we can consider the stable non-Gaussian EWMA model (stable EWMA). Suppose that the conditional distribution of the returns vector $z_{t+1} = \left[z_{1,t+1}, ..., z_{n,t+1} \right]'$ is α-stable sub-Gaussian (α∈(1,2)). Thus, for any time t, the centered vector of returns is given by

$$\tilde{z}_{t+1} = z_{t+1} - \mu_{t+1} = \Sigma_{t+1/t}\sqrt{B_{t+1}}G_{t+1}$$

where $B_{t+1} \sim S_{\alpha/2}\left(2\left(\cos\left(\frac{\pi\alpha}{4}\right)\right)^{2/\alpha}, 1, 0\right)$ is a stable subordinator independent of Gaussian vector $G_{t+1} = [G_{1,t+1}, \ldots, G_{n,t+1}]' \sim N(\mathbf{0}, \mathbf{I})$ with identity covariance matrix. In addition, $\sqrt{B_{t+1}}\, G_{t+1}$ t=0,…,T are i.i.d. α-stable sub-Gaussian vectors, where the components $\varepsilon_{i,t+1} = \sqrt{B_{t+1}} G_{i,t+1}$ are $S_\alpha(1,0,0)$ distributed, while the entries of dispersion matrix $Q_{t+1/t} = \left[\sigma^2_{ij,t+1/t}\right] = \Sigma_{t+1/t}\Sigma'_{t+1/t}$ are generated as follows:

$$\sigma^p_{ii,t+1/t} = E_t\left(\left|\tilde{z}_{i,t+1}\right|^p\right) A(p) = \lambda\sigma^p_{ii,t/t-1} + (1-\lambda)A(p)\left|\tilde{z}_{i,t}\right|^p, \quad (5)$$

$$\sigma^p_{i\oplus j,i\oplus j,t+1/t} = E_t\left(\left|\tilde{z}_{i,t+1} + \tilde{z}_{j,t+1}\right|^p\right) A(p) = \lambda\sigma^2_{i\oplus j,i\oplus j,t/t-1} + (1-\lambda)A(p)\left|\tilde{z}_{i,t} + \tilde{z}_{j,t}\right|^p, \quad (6)$$

$$\sigma^2_{ij,t+1/t} = \frac{\sigma^2_{i\oplus j,i\oplus j,t+1/t} - \sigma^2_{jj,t+1/t} - \sigma^2_{ii,t+1/t}}{2}, \quad (7)$$

where $A(q) = \frac{\Gamma\left(1-\frac{q}{2}\right)\sqrt{\pi}}{2^q\Gamma\left(1-\frac{q}{\alpha}\right)\Gamma\left(\frac{q+1}{2}\right)}$, $p \in (0,\alpha)$, λ is the decay factor that regulates the weighting on past covariation parameters. These assumptions are consistent with the structure of dispersion matrix of an α-stable sub-Gaussian vector. In particular, the scale parameter of *i*-th return, $\sigma_{ii,t+1/t}$ and the stable covariation parameter, $\sigma^2_{ij,t+1/t}$, are estimated by

$$\sigma_{ii,t+1/t} \approx \left((1-\lambda)A(p)\sum_{k=0}^{K}\lambda^{K-k}\left|\tilde{z}_{i,t-K+k}\right|^p\right)^{1/p} \text{ and}$$

$$\sigma^2_{ij,t+1/t} = \frac{\left((1-\lambda)A(p)\sum_{k=0}^{K}\lambda^{K-k}\left|\tilde{z}_{i,t-K+k} + \tilde{z}_{j,t-K+k}\right|^p\right)^{2/p} - \sigma^2_{jj,t+1/t} - \sigma^2_{ii,t+1/t}}{2},$$

where for any given tolerance level $tl = (1-\lambda)\sum_{k=K+1}^{\infty}\lambda^k$, we can determine the number of useful observations $K = \frac{\log(tl)}{\log(\lambda)}$ as per the RiskMetrics model. Thus, considering a tolerance level tl=0.001 and a decay factor λ=0.97, we obtain that $K \approx 228$. Under these assumptions, the (1-θ)% VaR in the period [t,t+1] is obtained by

$$VaR_{\theta,t+1/t}(z_{(p),t+1}) = k_{1-\theta,\alpha}\sigma_{(p),t+1/t},$$

where $k_{1-\theta,\alpha}$ is the corresponding percentile of the standardized α-stable $S_\alpha(1,0,0)$ and $\sigma_{(p),t+1/t} = \sqrt{w'Q_{t+1/t}w}$ is the forecasted volatility of portfolio $z_{(p),t+1}$. In addition, if we assume an unconditional model where the vectors $\tilde{z}_{t+s}$ are i.i.d. α-stable sub-Gaussian dis-

tributed, then the aggregated process $Z_{t+T} = \sum_{s=1}^{T} \tilde{z}_{t+s}$ is α-stable sub-Gaussian distributed with dispersion that follows the time rule $\tilde{Q}_{t+T/t} = T^{2/\alpha} Q_{t+1/t}$. Thus, even in this case, we can predict the (1-θ)% VaR over the period [*t*,*t*+*T*] with the following approximating time rule:

$$VaR_{\theta,t+T/t} \approx (T)^{1/\alpha} VaR_{\theta,t+1/t}, \tag{8}$$

whose predictability power will be empirically tested in the next empirical analysis.

2.3. Stable asymmetric model

In order to take into account the asymmetry of stable distributions, we can consider the following three-fund separation model of conditional centered returns:

$$\tilde{z}_{t+1} = z_{t+1} - \mu_{t+1} = b_{t+1} Y_{t+1} + \Sigma_{t+1/t} \sqrt{B_{t+1}} G_{t+1},$$

where $\mu_{t+1} = E(z_{t+1})$, the factor $Y_{t+1} \sim S_\alpha(\sigma_{Y_{t+1}}, \beta_{Y_{t+1}}, 0)$ is an α-stable asymmetric (i.e. $\beta_{Y_{t+1}} \neq 0$) centered index return with dispersion and the skewness respectively equal to $\sigma_{Y_{t+1}}$ and $\beta_{Y_{t+1}}$. Besides the residual random vector $\tilde{z}_{t+1} - b_{t+1} Y_{t+1} = \Sigma_{t+1/t} \sqrt{B_{t+1}} G_{t+1}$ is independent of factor Y_{t+1} and it is conditional α-stable sub-Gaussian distributed (as the above Stable EWMA model) with zero mean and dispersion matrix $Q_{t+1/t} = \left[\sigma^2_{ij,t+1/t} \right] = \Sigma_{t+1/t} \Sigma'_{t+1/t}$.

Thus, the primary returns and the dispersion matrix of residuals evolve as follows:

$$\begin{aligned} \tilde{z}_{i,t+1} &= b_{i,t+1} Y_{t+1} + \sigma_{ii,t+1/t} \varepsilon_{i,t+1} = \left(\sigma^\alpha_{ii,t+1/t} + \left| b_{i,t+1} \sigma_{Y_{t+1}} \right|^\alpha \right)^{1/\alpha} X_{i,t+1}, \\ \sigma^p_{ii,t+1/t} &= E_t \left(\left| \tilde{z}_{i,t+1} \right|^p \right) A(p) = \lambda \sigma^p_{ii,t/t-1} + (1-\lambda) A(p) \left| \tilde{z}_{i,t} - b_{i,t} Y_t \right|^p \end{aligned} \tag{9}$$

$$\begin{aligned} \sigma^p_{i\oplus j, i\oplus j, t+1/t} &= E_t \left(\left| \tilde{z}_{i,t+1} + \tilde{z}_{j,t+1} - (b_{i,t+1} + b_{j,t+1}) Y_{t+1} \right|^p \right) A(p) = \\ &= \lambda \sigma^2_{i\oplus j, i\oplus j, t/t-1} + (1-\lambda) A(p) \left| \tilde{z}_{i,t} + \tilde{z}_{j,t} - (b_{i,t} + b_{j,t}) Y_t \right|^p \\ \sigma^2_{ij,t+1/t} &= \frac{\sigma^2_{i\oplus j, i\oplus j, t+1/t} - \sigma^2_{jj,t+1/t} - \sigma^2_{ii,t+1/t}}{2}, \end{aligned} \tag{10}$$

where $p \in (0,\alpha)$, and $\varepsilon_{i,t} = \sqrt{B_{t+1}} G_{i,t+1} \sim S_\alpha(1,0,0)$. The vector $b_t = \left[b_{1,t}, ..., b_{n,t} \right]'$ is estimated considering the OLS estimator then $\hat{b}_i = \frac{\sum_{k=1}^{N} Y^{(k)} \tilde{z}_i^{(k)}}{\sum_{k=1}^{N} \left(Y^{(k)} \right)^2}$,

i=1,...,*n.* In our empirical comparison, we have observed better performance when the vector $b_{t+1} = \left[b_{1,t+1/t}, ..., b_{n,t+1/t} \right]'$ changes over the time, following the relation

$$b_{i,t+1/t} = \frac{\sum_{k=0}^{K} Y_{t-K+k} \tilde{z}_{i,t-K+k}}{\sum_{k=0}^{K} \left(Y_{t-K+k} \right)^2} = \frac{\hat{b}_{i,t+1/t}^{(1)}}{\hat{b}_{t+1/t}^{(2)}}; \qquad \text{i=1,...,}n,$$

where $\hat{b}_{i,t+1/t}^{(1)} = \hat{b}_{i,t/t-1}^{(1)} + Y_t \tilde{z}_{i,t} - Y_{t-K-1} \tilde{z}_{i,t-K-1}$ and $\hat{b}_{t+1/t}^{(2)} = \hat{b}_{t/t-1}^{(2)} + \left(Y_t \right)^2 - \left(Y_{t-K-1} \right)^2$. On the other hand, recent studies (see Kurz-Kim et al., 2005) have proved the opportunity of using other estimators for the vector b_t. The scale parameter of *i*-th residual, $\sigma_{ii,t+1/t}$ is defined by:

$$\sigma_{ii,t+1/t} = \left(E_t \left(\left| \tilde{z}_{i,t+1} - b_{i,t+1} Y_{t+1} \right|^p \right) A(p) \right)^{1/p} \approx \left(\left(1-\lambda\right) A(p) \sum_{k=0}^{K} \lambda^{K-k} \left| \tilde{z}_{i,t-K+k} - b_{i,t-K+k} Y_{t-K+k} \right|^p \right)^{1/p}$$

and the time *t*+1 stable covariation parameter between the *i*-th and the *j*-th residual is defined by $\sigma_{ij,t+1/t}^2 = \frac{\sigma_{i\oplus j,i\oplus j,t+1/t}^2 - \sigma_{jj,t+1/t}^2 - \sigma_{ii,t+1/t}^2}{2}$, where

$$\sigma_{i\oplus j,i\oplus j,t+1/t}^2 \approx \left(\left(1-\lambda\right) A(p) \sum_{k=0}^{K} \lambda^{K-k} \left| \tilde{z}_{i,t-K+k} + \tilde{z}_{j,t-K+k} - (b_{i,t-K+k} + b_{j,t-K+k}) Y_{t-K+k} \right|^p \right)^{1/p}$$

.

Under these assumptions, the forecasted (1-θ)% Value at Risk of portfolio $z_{(p),t+1} = \sum_{i=1}^{n} w_i \tilde{z}_{i,t+1} = w' \tilde{z}_{t+1}$ in the period [*t,t*+1], is given by the corresponding percentile of the α-stable distribution $S_\alpha (\sigma_{(p),t+1/t}, \beta_{(p),t+1/t}, 0)$, where

$$\sigma_{(p),t+1/t} = \left(\left| w' Q_{t+1/t} w \right|^{\alpha/2} + \left| w' b_{t+1} \sigma_{Y_{t+1}} \right|^{\alpha} \right)^{1/\alpha}$$

is the volatility forecast and

$$\beta_{(p),t+1/t} = \frac{\left| w' b_{t+1} \sigma_{Y_{t+1}} \right|^{\alpha} \beta_{Y_{t+1}} \operatorname{sgn}(w' b_{t+1})}{\left(\left| w' Q_{t+1/t} w \right|^{\alpha/2} + \left| w' b_{t+1} \sigma_{Y_{t+1}} \right|^{\alpha} \right)}$$

is the skewness forecast. Moreover, to take into account the evolution of the index *Y* and its fixed memory, we assume that the dispersion parameter σ_{Y_t} follows the recursive formula

$$\sigma_{Y_t}^p = \frac{A(p) E_t \left(\left| Y_{t+1} \right|^p \right)}{g\left(\alpha_Y, \beta_{Y_t}, p \right)} \approx \frac{A(p) \sum_{k=0}^{K} \left| Y_{t-K+k} \right|^p}{K g\left(\alpha_Y, \beta_{Y_t}, p \right)} \approx \sigma_{Y_{t-1}}^p + A(p) \frac{\left| Y_t \right|^p - \left| Y_{t-K-1} \right|^p}{K g\left(\alpha_Y, \beta_{Y_t}, p \right)}, \qquad (11)$$

where

$$g\left(\alpha_Y,\beta_{Y_t},p\right)=\left(1+\beta_{Y_t}^2\left(\tan^2\left(\frac{\alpha_Y\pi}{2}\right)\right)\right)^{\frac{p}{2\alpha_Y}}\cos\left(\frac{p}{2\alpha_Y}\arctan\left(\beta_{Y_t}\left(\tan\left(\frac{\alpha_Y\pi}{2}\right)\right)\right)\right).$$

A detailed analysis of stable distribution properties can be found in Samorodnitsky and Taqqu (1994). The distribution of aggregated process $Z_{t+T}=\sum_{s=1}^{T}\tilde{z}_{t+s}$ is not known. However, if we suppose that the vectors $\tilde{z}_{t+s}=b_{t+s}Y_{t+s}+\Sigma_{t+s/t+s-1}\varepsilon_{t+s}$ (s=1,…,T) are i.i.d. α-stable distributed where $Q_{t+1/t}=\Sigma_{t+1/t}\Sigma'_{t+1/t}=\Sigma_{t+s/t+s-1}\Sigma'_{t+s/t+s-1}$ is the dispersion matrix of the α-stable sub-Gaussian vector $\Sigma_{t+s/t+s-1}\varepsilon_{t+s}$ that is independent of $Y_{t+s}\square S_\alpha(\sigma_{Y_{t+s}},\beta_{Y_{t+s}},0)$ (s=1,…,T), then $Z_{t+T}=\sum_{s=1}^{T}\tilde{z}_{t+s}$ is itself α-stable distributed. Under these assumptions the dispersion $\Theta_{t+T/t}$ at time *t* of the aggregated process of residuals $Z_{t+T}-\sum_{s=1}^{T}b_{t+s}Y_{t+s}$ follows the time rule $\Theta_{t+T/t}=T^{2/\alpha}Q_{t+1/t}$. In particular, when we further assume that the parameters α, $\beta_{Y_t},\sigma_{Y_t},b_t$ are constant over the time, then the corresponding (1-θ) percentile of aggregated portfolio $Z_{(p),t+T}=w'Z_{t+T}$ can be approximated with the following time rule:

$$VaR_{\theta,t+T/t}\approx\left(T\right)^{1/\alpha}VaR_{\theta,t+1/t}.$$

Clearly, this time rule cannot be applied to the above autoregressive stable model (see Lamantia et al., 2006).

3. A first empirical analysis on VaR models based on real data

This section presents an analysis through backtesting in order to assess the reliability of the VaR models previously proposed. We use some of the most representative index returns of the international market whose values have been converted in USD with the relative exchange rates. In particular we examine the Value at Risk of several portfolios composed of 10 index-daily returns: DAX 100 Performance, CAC 40, FTSE all share, Reuters Commodities, Nikkei 500, Brent Crude Physical, Corn No2 Yellow cents, Dow Jones Industrials, Goldman Sachs Commodity, S&P 500.

First of all, we discuss on how to estimate the parameters of each model. Thus, we consider unconditional series of daily returns between January 2, 1991 and August 19, 1994, for a total of 948 observations to compute the distributional parameters of each model (see Table 1). Part of this historical data (during the period of November 15, 1993-August 19, 1994) is also used to estimate the dispersion matrixes of the different models because this period is prior to the period used to compute and compare ex-post the different VaR forecasts. Then over a period of 769 working days (till January 30, 1998), we verify the hypothesis that the VaR, computed at the beginning of each period, correctly forecasts the realization of the actual profit/loss occurred at the end of the period. In particular we calculate the interval forecasts considering θ=95% and θ=99%. We propose three different methods for evaluating Value at Risk estimates of 25 random portfolios:

a) a basic backtest method consisting in testing if the average coverage of the VaR is equal to the nominal coverage;

b) the unconditional coverage test proposed by Kupiec (1995), Lopez (1998), Christoffersen (1998);

c) the interval forecast method proposed by Christoffersen (1998) in order to test the conditional coverage.

Table 1

This table summarizes the parameter estimates of different models. Thus, we consider: the maximum likelihood estimates (MLE) of the stable distribution parameters and of Student's *t* degrees of freedom, the symmetric and asymmetric optimal stable parameters *p* that minimize the average of distance between the moment dispersion estimator and MLE of dispersion. Observe that for DAX30 we compute only the Stable parameter estimates because it was only used as market portfolio in the asymmetric stable model. All parameters are computed on series of daily returns between January 2, 1991 and August 19, 1994 for a total of 948 observations.

ASSETS	STABLE DISTRIBUTIONS						Student's Degrees of freedom
	Maximum Likelihood Estimates				SEWMA	Stable Model with asymmetry	
	Index of stability α	Skewness β	Dispersion γ	Mean μ	*Optimal p*	*Optimal p*	*MLE v*
BRENT CRUDE	1.6383	-0.0004	9.79E-03	-1.35E-04	0.855	0.824	7.00
CAC 40	1.8704	-0.3033	7.71E-03	1.62E-04	0.54	0.525	4.55
CORN, NO.2	1.6757	-0.1966	6.61E-03	-2.07E-04	0.635	0.611	4.61
DAX 30	1.7557	-0.1622	3.12E-03	1.68E-04	//	//	//
DAX 100	1.7489	-0.1803	6.75E-03	3.18E-04	0.697	0.662	3.54
DOW JONES IND.	1.7559	0.0558	4.18E-03	3.68E-04	0.25	0.29	6.95
FTSE ALL SHARE	1.8505	-0.0126	6.18E-03	2.48E-04	0.774	0.745	5.64
GOLDMAN SACHS	1.7051	0.0096	4.53E-03	-1.65E-04	0.773	0.789	20.46
NIKKEI 500	1.6922	-0.0228	7.97E-03	1.32E-04	0.263	0.245	60.34
S&P 500	1.7123	0.1482	3.98E-03	4.30E-04	0.429	0.478	11.44
REUTERS COMM.	1.795	-0.1346	5.53E-03	-6.71E-05	0.287	0.323	3.77

3.1. Parameter estimation with different models

Among the elliptical EWMA models with finite variance, we compare the Gaussian and the Student's *t* EWMA models. When we assume that the return vector $z_s = \left[z_{1,s}, ..., z_{n,s} \right]'$ follows a Gaussian EWMA model, the decay factor λ_i of the *i*-th component of the return vector is estimated by using the same procedure of RiskMetrics, i.e. we evaluate for any series

$$\lambda_i = \arg\left(\min_{\bar{\lambda}} \sum_{t=1}^{T} \left| z_{i,t}^2 - \sigma_{ii,t/t-1}^2 \left(\bar{\lambda} \right) \right|^2 \right).$$

Then, the optimal parameter λ of the vector is defined as

$$\hat{\lambda} = \sum_{t=1}^{n} \phi_i \lambda_i \,, \tag{12}$$

where $\phi_i = \frac{\vartheta_i}{\sum_{k=1}^{n} \vartheta_k}$ and $\vartheta_i = \frac{1}{\sqrt{\min_{\bar{\lambda}} \frac{1}{T} \sum_{t=1}^{T} \left| z_{i,t}^2 - \sigma_{ii,t/t-1}^2 \left(\bar{\lambda} \right) \right|^2}}$.

An elliptical distribution that has often been used in the literature in order to model the leptokurtic behavior of conditional returns (see, among others, Embrechts et al., 2003) is the multivariate Student's *t* distribution $\mathbf{MV}\text{-}t_v(\mathbf{0},\mathbf{Q})$ with v>2 degrees of freedom. Therefore, we can assume that the return vector $z_s = \left[z_{1,s},...,z_{n,s} \right]'$ follows an EWMA model with conditional Student's *t* distributed returns and v>2 degrees of freedom. Under this assumption every return $z_{i,s}$ admits the following conditional density function:

$$t\left(x/v,\sigma_{ii,s/s-1}\right) = \frac{\Gamma\left(\frac{v+1}{2}\right)}{\left((v-2)\pi\right)^{1/2}\Gamma\left(\frac{v}{2}\right)\sigma_{ii,s/s-1}}\left(1+\frac{x^2}{(v-2)\sigma^2_{ii,s/s-1}}\right)^{-\frac{v+1}{2}}.$$

Besides, the vector of returns can be represented as follows

$$z_{s+1} = \Sigma_{v,s+1/s}\frac{\sqrt{v}}{\sqrt{S_{s+1}}}G_{s+1}, \tag{13}$$

where the vectors $\frac{\sqrt{v}}{\sqrt{S_{s+1}}}G_{s+1}$ are i.i.d. Student's *t* distributed, $S_{s+1} \sim \chi^2(v)$ is a chi-square with v degrees of freedom independent of Gaussian vector $G_{s+1} \sim N(\mathbf{0},\mathbf{I})$. Thus, the variance covariance matrix of the above conditional Student's *t* is given by:

$$Q_{t+1/t} = \left[\sigma^2_{ij,t+1/t} \right] = \frac{v}{v-2}\Sigma_{v,t+1/t}\Sigma'_{v,t+1/t}$$

and it evolves as in formulas (1) and (2).

3.1.1. Parameter estimation of Student's t EWMA model

In the EWMA model with conditional Student's *t* distributed returns, we need to estimate two parameters: the decay factor λ and the degrees of freedom v. In the following empirical analysis we compare the impact of two different methods to estimate v and λ. If we estimate the degrees of freedom v_i of each conditional return with the maximum likelihood method, we obtain different values v_i>2 for different return series (see Table 1). Then, we can assume that the parameters λ and v are a particular mean of the parameters λ_i, v_i evaluated for every distinct series of data. Hence, the optimal parameter λ is estimated using the above RiskMetrics procedure (12), and, similarly, we can define the optimal parameters v as: $\hat{v} = \sum_{t=1}^{n}\phi_i v_i$, where ϕ_i i=1,…,*n* are defined as in formula (12). Under these assumptions the degrees of freedom v and the corresponding percentile, $k_{1-\theta,v}$, of the standardized Student's *t* distribution with v>2 degrees of freedom $t_v(0,1)$ change over the time. This issue ensures a further flexibility in the prediction of the future percentile

$$VaR_{\theta,t+1/t}(z_{(p),t+1}) = k_{1-\theta,v}\sigma_{(p),t+1/t}.$$

Alternatively, for computational simplicity, we fix the parameters v and λ. In particular, we observe that λ=0.9718 is the mean of the λ computed as above and the corresponding 95% interval of confidence is (0.94, 0.99). Thus, we adopt the 10^{-2} approximation of λ, that is λ=0.97. Moreover, in order to use a common value v of degrees of freedom we could fix $v = \left\lceil \min_i v_i \right\rceil$=4 (where $\lceil . \rceil$ points out the integer part plus 1). As a matter of fact, market return series generally

present an asymptotic dependence on extremes (see Breymann, et al, 2003). The Student's *t*-copula with the minimum of the degrees of freedom *v* represents a tool for modeling this phenomenon because stronger dependencies on the tails correspond to lower degrees of freedom (see, among others, Embrechts et al., 1997; Nelsen, 1999; Breymann et al., 2003).

3.1.2. Parameter estimation of stable EWMA model

In order to choose opportune parameters of the stable Paretian models, we first estimate the stable distribution parameters by maximizing the likelihood function (see Rachev and Mittnik, 2000). Once parameters *α, γ, β, δ* are evaluated, it is possible to determine the VaR on distinct assets using tabled percentiles of stable distributions or some particular software. In Table 1, we display, among other distributional parameters, the parameter estimates of a stable density fit based on the sample of some international indexes. We use the FinAnalytica software[1] to estimate the parameters *α, γ, β* and *δ*. All the assets present an index of stability *α* that is lower than two according to other researches on the unconditional distributions of financial returns. Also the skewness parameter *β* results significantly different from zero, showing the typical negative skewness that stretches the tails on the negative returns. Observe that we obtain different indexes of stability α_i for each time series (see Table 1). Thus, as for the degrees of freedom of *t*-distributions, we will propose two different ways to evaluate a common index of stability *α*. On the other hand, in the above stable EWMA model we need to estimate three parameters: the index of stability *α*, the decay factor *λ* and parameter *p*. When vector *z* is unconditional *α*-stable sub-Gaussian distributed with dispersion matrix $V = \left[v_{ij}^2 \right]$, the rate of convergence of v_{ij}^2 and v_{jj}^2 will be faster if *p* is as small as possible (about the rate of convergence of stable laws and of L^p norms of stable laws, see Rachev, 1991). Considering that the impact in the dispersion valuation of a portfolio is substantially determined by the dispersion of the singular components (see Ziemba and Mulvey, 1999), then we compute *p* considering only the optimal values of the singular components. Theoretically, the optimal *p* must be close to zero for stable distributions because the rate of convergence for stable non- Gaussian law of L^p norm (i.e. $\left(E\left(\left| \tilde{z}_i \right|^p \right) \right)^{1/p}$) is faster if *p* is lower. However, if we approximate $\tilde{z}_j$ with a stable distribution, the optimal $p \in (0,\alpha)$ depends on the historical series of observations $\left\{ \tilde{z}_{j,(k)} \right\}_{k=1}^{N}$. Thus, we consider the optimal p_j that minimizes the average of absolute deviation between

$$\hat{\sigma}_{ii,t+1/t}(p) = \left((1-\lambda) A(p) \sum_{k=0}^{K} \lambda^{K-k} \left| \tilde{z}_{i,t-K+k} \right|^p \right)^{1/p}$$ (that we call *moment dispersion estimator*) and the maximum likelihood estimate $\overline{v}_{jj}$ of dispersion v_{jj}, i.e.:

$$\hat{p}_j = \arg\left(\min_p \frac{1}{T} \sum_{t=1}^{T} \left| \hat{\sigma}_{jj,t/t-1}(p) - \overline{v}_{jj} \right| \right) \qquad \text{j=1,...,n.} \tag{14}$$

In Table 1, we report optimal $\hat{p}_j$ of daily return series between January 2, 1991 and August 19, 1994. In the paper, we adopt the common parameter $p = \frac{1}{n} \sum_{j=1}^{n} \hat{p}_j$. As far as the other parameters are concerned, we observe that it could be convenient to fix the parameters *α* and *λ* for large portfolios, as per the RiskMetrics model (that suggests *λ*=0.94 for daily historical series) or

[1] This software has been developed by FinAnalytica Inc. http//www.finanalytica.com.

the EWMA model with conditional *t*-distributed returns (where we consider λ=0.97 and $v=\lceil \min v_i \rceil$). Even in this case, we observe that the average of λ computed as specified below, is λ=0.9734 and the corresponding 95% interval of confidence is (0.94, 1). Thus we adopt the 10^{-2} approximation of λ that is again λ=0.97 and we assume $\alpha = \frac{1}{n}\sum_{j=1}^{n}\alpha_j$. Moreover, when we fix the parameters α and λ, the stable EWMA model presents the same computational complexity as the RiskMetrics one (after parameters estimation), as confirmed by empirical analysis. However, as for the RiskMetrics model, we could estimate the decay factors λ_i by minimizing the root mean squared prediction error (RMSE) on the historical series of data. Substantially, the procedure is the same as in RiskMetrics, i.e. we evaluate for any series

$$\lambda_i = \arg\left(\min_{\bar{\lambda}} \frac{1}{T}\sum_{t=1}^{T} \left| A(p)\left|\tilde{z}_{i,t}\right|^p - \sigma_{ii,t/t-1}^p\left(\bar{\lambda}\right)\right|^2 \right). \quad (15)$$

There has been an extensive discussion among academics and practitioners on which error measure to use when assessing post-sample prediction (see Ahlburg, 1992; Armstrong and Collopy, 1992; Fildes, 1992). In the following empirical analysis we use the root mean squared prediction error regardless if similar results are obtained also with other risk measures. Thus, we solve optimization problem (15) by discretizing $\bar{\lambda}_i$ with the same steps $\Delta\bar{\lambda}_i \square 0.01$ for every *i*. The optimal parameters λ, α are defined as:

$$\hat{\lambda} = \sum_{i=1}^{n}\phi_i\lambda_i \quad \text{and} \quad \hat{\alpha} = \sum_{i=1}^{n}\phi_i\alpha_i\,, \quad (16)$$

where α_i is estimated with the maximum likelihood method $\phi_i = \frac{\vartheta_i}{\sum_{k=1}^{n}\vartheta_k}$ and $\vartheta_i = \frac{1}{\sqrt{\min_{\bar{\lambda}} \frac{1}{T}\sum_{t=1}^{T}\left| A(p)\left|\tilde{z}_{i,t}\right|^p - \sigma_{ii,t/t-1}^p\left(\bar{\lambda}\right)\right|^2}}$. Under these assumptions the index of stability α and the corresponding percentile, $k_{1-\theta,\alpha}$, of the standardized α-stable $S_\alpha(1,0,0)$ change over the time. This issue ensures a further flexibility in the prediction of the future VaR.

Table 2

This table summarizes the backtesting results of EWMA with Gaussian and Student's *t* distributions, SEWMA and stable asymmetric models for ϑ=99%.

	Student's *t*-distribution		Gaussian Distribution	STABLE DISTRIBUTIONS		
	v=4, λ=0.97	Varying v and λ	Varying λ	α=1.7444 p=0.55, λ=0.97	Varying α and λ p=0.55	Asymmetric varying λ α=1.7557 p=0.549
	VaR 1%	VaR 1%	VaR 1%	VaR 1%	VaR 1%	VaR 1%
Port.1	0.00390	0.01300	0.02341	0.00910	0.00910	0.01170
Port.2	0.00390	0.01691	0.02081	0.01040	0.01040	0.01170
Port.3	0.00260	0.01691	0.01951	0.00780	0.00780	0.00910
Port.4	0.00780	0.01821	0.02601	0.00910	0.00910	0.00910

Table 2 (continuous)

Port.5	0.00130	0.01170	0.02211	0.00780	0.00780	0.00910
Port.6	0.00260	0.01300	0.01951	0.00650	0.00650	0.00520
Port.7	0.00000	0.01951	0.02731	0.00910	0.00910	0.01170
Port.8	0.00390	0.01170	0.02211	0.00650	0.00650	0.01040
Port.9	0.00390	0.01560	0.02731	0.00780	0.00650	0.01040
Port.10	0.00520	0.02341	0.03251	0.00910	0.01170	0.01170
Port.11	0.00000	0.01170	0.01951	0.00650	0.00650	0.00650
Port.12	0.00260	0.01040	0.02081	0.00780	0.00650	0.00910
Port.13	0.00650	0.01430	0.02341	0.00910	0.00780	0.01170
Port.14	0.00390	0.01430	0.02471	0.00910	0.00910	0.01170
Port.15	0.00130	0.01560	0.02081	0.00650	0.00780	0.00780
Port.16	0.00390	0.02211	0.02861	0.00910	0.00910	0.01170
Port.17	0.00130	0.00910	0.01560	0.00650	0.00520	0.01040
Port.18	0.00260	0.01300	0.02211	0.00910	0.01040	0.01040
Port.19	0.00260	0.01821	0.02341	0.00910	0.00910	0.01040
Port.20	0.00000	0.00520	0.01821	0.00130	0.00130	0.00520
Port.21	0.00390	0.01691	0.02081	0.01170	0.01170	0.01040
Port.22	0.00390	0.01821	0.02601	0.00780	0.00910	0.00780
Port.23	0.00650	0.02471	0.03251	0.00780	0.00910	0.01300
Port.24	0.00390	0.02081	0.02601	0.01040	0.01040	0.01170
Port.25	0.00390	0.01821	0.02991	0.00650	0.00650	0.01170
Average	0.00328	0.01571	0.02372	0.00806	0.00817	0.00999

3.1.3. The Parameter Estimation of stable asymmetric model

In the model with asymmetrically distributed returns we first estimate the maximum likelihood stable parameters α_Y, σ_Y, β_Y of index Y (that is DAX30 in our empirical analysis). However, we assume that the dispersion parameter σ_{Y_t} evolves as in formula (11) where the maximum likelihood estimate $\hat{\sigma}_{Y_0}$ is used as the first value of that recursive formula. Besides, we assume that $\beta_{Y_t} = \mathrm{sgn}(\beta_Y)$ such that some portfolios could present large skewness (i.e. $\beta_{(p),t+1/t} \neq \pm 1$) (because in this model $\left|\beta_{(p),t+1/t}\right| \leq \left|\beta_{Y_t}\right| = 1$). With respect to the parameter α, we choose $\alpha=\alpha_Y$, equal to the index of stability estimated for the index Y. The vector $b_t = \left[b_{1,t},...,b_{n,t}\right]'$ is estimated considering the OLS estimator then $\hat{b}_i = \dfrac{\sum_{k=1}^{N} Y^{(k)} \tilde{z}_i^{(k)}}{\sum_{k=1}^{N} \left(Y^{(k)}\right)^2}$, i=1,…,*n*. However, after a first empirical comparison we observe better performance when we fix the memory of the process and the vector $b_{t+1} = \left[b_{1,t+1/t},...,b_{n,t+1/t}\right]'$ changes over the time, following the relation:

$$b_{i,t+1/t} = \frac{\sum_{k=0}^{K} Y_{t-K+k}\tilde{z}_{i,t-K+k}}{\sum_{k=0}^{K}\left(Y_{t-K+k}\right)^2} = \frac{\hat{b}^{(1)}_{i,t+1/t}}{\hat{b}^{(2)}_{t+1/t}}; \qquad \text{i=1,…,}n, \tag{17}$$

where $\hat{b}^{(1)}_{i,t+1/t} = \hat{b}^{(1)}_{i,t/t-1} + Y_t\tilde{z}_{i,t} - Y_{t-K-1}\tilde{z}_{i,t-K-1}$ and $\hat{b}^{(2)}_{t+1/t} = \hat{b}^{(2)}_{t/t-1} + \left(Y_t\right)^2 - \left(Y_{t-K-1}\right)^2$. As far as the vector b_t is concerned, we assume that it changes over the time, following relation (17). As for the sub-Gaussian model we need to estimate other two parameters: the decay factor λ and parameter p. We evaluate for any series the decay factor:

Table 3

This table summarizes the backtesting results of EWMA with Gaussian and Student's *t* distributions, SEWMA and stable asymmetric models forϑ=95%..

	Student's *t*-distribution		Gaussian Distribution	STABLE DISTRIBUTIONS		
	v=4, λ=0.97	Varying *v* and λ	Varying λ	α=1.7444 p=0.55, λ=0.97	Varying α and λ p=0.55	Asymmetric varying λ α=1.7557 p=0.549
	VaR 5%	VaR 5%	VaR 5%	VaR 5%	VaR 5%	VaR 5%
Port.1	0.02731	0.05332	0.05852	0.06112	0.06112	0.04421
Port.2	0.02601	0.05072	0.05852	0.06242	0.06372	0.05852
Port.3	0.02471	0.05072	0.05852	0.06632	0.06632	0.04811
Port.4	0.03511	0.05722	0.06762	0.06632	0.06242	0.06632
Port.5	0.03381	0.05462	0.05852	0.06502	0.05982	0.06242
Port.6	0.02861	0.04811	0.05852	0.06502	0.05982	0.04811
Port.7	0.03771	0.06502	0.07282	0.06242	0.06502	0.06372
Port.8	0.02861	0.05332	0.06632	0.06632	0.06502	0.06892
Port.9	0.03381	0.06242	0.07412	0.06632	0.06502	0.05072
Port.10	0.03901	0.06502	0.07932	0.06242	0.06502	0.06502
Port.11	0.02471	0.05072	0.05332	0.05982	0.06112	0.04551
Port.12	0.02471	0.05072	0.05722	0.06632	0.06112	0.04681
Port.13	0.03251	0.05592	0.06632	0.05852	0.05722	0.04941
Port.14	0.03381	0.05592	0.06372	0.05982	0.05982	0.04811
Port.15	0.02731	0.04681	0.05722	0.06112	0.05982	0.05072
Port.16	0.03511	0.06242	0.06762	0.06632	0.06372	0.04941
Port.17	0.02731	0.05852	0.06632	0.07672	0.07412	0.06112
Port.18	0.02861	0.04551	0.05462	0.05462	0.05722	0.05202
Port.19	0.02731	0.05072	0.05332	0.06242	0.06242	0.04551
Port.20	0.02471	0.05462	0.06372	0.06632	0.06372	0.07022
Port.21	0.02731	0.04681	0.05202	0.06242	0.05852	0.04551
Port.22	0.02731	0.05592	0.06242	0.06762	0.06632	0.05202
Port.23	0.04161	0.05982	0.06632	0.06242	0.06372	0.07022
Port.24	0.03511	0.05852	0.06632	0.07022	0.07022	0.05852
Port.25	0.03511	0.05462	0.06242	0.06372	0.06242	0.06372
Average	0.03069	0.05472	0.06263	0.06408	0.06299	0.05540

$$\lambda_i = \arg\left(\min_{\bar{\lambda}} \frac{1}{T} \sum_{t=1}^{T} \left| A(p) \left| \tilde{z}_{i,t} - b_{i,t} Y_t \right|^p - \sigma_{ii,t/t-1}^p \left(\bar{\lambda} \right) \right|^2 \right).$$

Thus, the optimal parameter λ is defined as $\hat{\lambda} = \sum_{i=1}^{n} \phi_i \lambda_i$, where $\phi_i = \frac{\vartheta_i}{\sum_{k=1}^{n} \vartheta_k}$ and $\vartheta_i = \frac{1}{\sqrt{\min_{\bar{\lambda}} \frac{1}{T} \sum_{t=1}^{T} \left| A(p) \left| \tilde{z}_{i,t} - b_{i,t} Y_t \right|^p - \sigma_{ii,t/t-1}^p \left(\bar{\lambda} \right) \right|^2}}$. Analogously, to the stable EWMA model and for the same reasons, we assume that parameter p is equal to the mean of optimal p_i that minimizes the average of distance between the moment dispersion estimator of residuals $\tilde{z}_{i,t} - b_{i,t} Y_t$ and its maximum likelihood stable estimate (see Table 1).

3.2. The basic backtest method

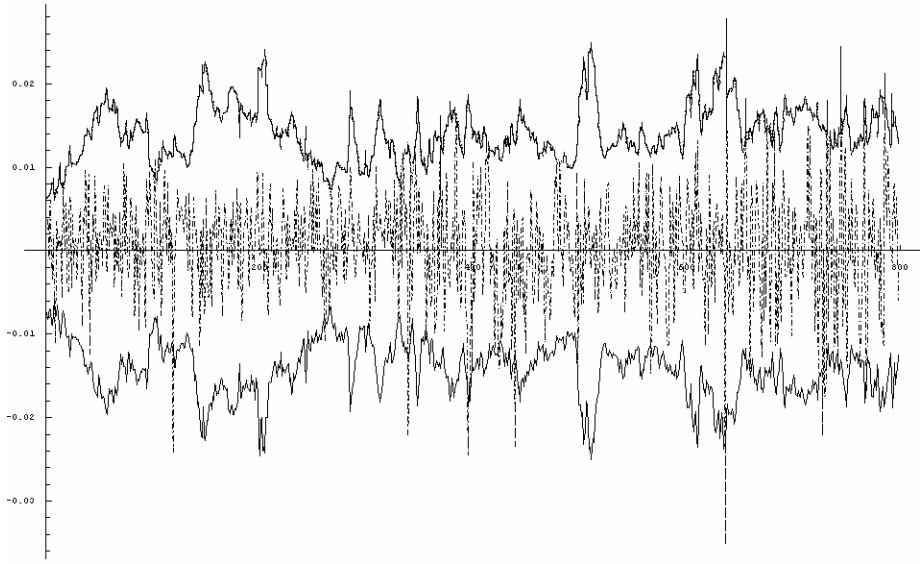

Fig. 1. VaR Bands of SEWMA model with varying α. The lowest band represents the forecast 5% percentiles, whilst upper band represents the forecast 95% percentiles. The inner path shows sample data of the 4th portfolio

In the first backtest analysis proposed, we have determined how many times during the period taken into account the profits/losses fall outside the confidence interval. In particular, for θ=95% and θ=99%, the expected number of observations outside the confidence interval must not exceed respectively 5% and 1%.

Tables 2 and 3 show the results for the backtest for the two levels of confidence respectively.

The backtest has been carried out on some randomly selected portfolios. A first analysis demonstrates that the RiskMetrics model, under the hypothesis of normality of the conditional returns, underestimates the number of observations which falls outside the forecast interval. This effect is more evident when the percentiles are low and it confirms that the empirical distribution tails are fatter.

In Tables 2 and 3, we can compare the backtest results among the elliptical EWMA models and the stable asymmetric model for θ=95% and θ=99%. In view of this comparison, we assume that p=0.55 while the stable EWMA stability index α either varies over time as described above or it is constant α=1.7444. We observe that the results that we obtain varying or non-varying

α are not significantly different. Therefore, by a computational point of view, it is more convenient to use a fixed α. Similarly, we assume that the degrees of freedom v of an elliptical model with conditional t distributed returns could vary over the time as described above or they are constant with v=4. In this case, varying v improves much more the results. Recall that the formula for the portfolio's VaR bands is given by $VaR_{\theta,t+1/t} = \pm k_{1-\theta}\sigma_{(p),t+1/t}$ where $k_{1-\theta}$ is the percentile of the standardized elliptical family. In the stable asymmetric model, we use the DAX 30 series as index return Y and we fix the parameters β_Y, α and p of the model respectively with β_Y=-1, α=α_Y=1.7557 and p=0.549. This way, the stable asymmetric model appears as flexible as the stable EWMA model and presents better performances for higher percentiles. Therefore, if α and v vary over time, as described in the above models, the VaR forecasts are much more sensitive to the last information of the market.

However, we do not observe significant differences between the stable EWMA model with fixed or varying α, while important differences exist between the two Student's t models.

In particular, it seems that the Student's t model with varying v and the stable asymmetric model present better performance for higher percentiles, while all the stable models present better performance for smaller percentiles.

Figure 1 shows the typical representation of 1-day stable EWMA VaR forecasts considering that the index of stability α varies over the time. The lowest band represents the stable EWMA VaR forecasts while the upper band represents 95% percentile forecasts and the inner path shows the sample data.

Among the alternative models for the VaR calculation, stable models are more reliable than those generated from RiskMetrics, in particular for confidence interval θ=99%. The advantage of using stable models as an alternative to the normal one is reduced when the percentiles are higher than 5%. In this case, the percentage realized is almost equal to that expected, except for the stable sub-Gaussian model that overestimates the losses.

Even if the non-Gaussian models seem more complex than the RiskMetrics one, once we estimate the parameters of Table 1, there is no significant difference in computational time between each parametric model and the RiskMetrics one. Thus, we believe that these parametric models could be implemented for online VaR calculation because they show a high degree of efficiency and flexibility for large portfolios. This is a desirable goal, considering that with the advent of new technology and on-line trading, there has been a growth in the number of those investors who prefer "*tailoring*" their own portfolios.

3.3. Conditional and unconditional coverage tests

Under every distributional hypothesis and for every portfolio $z_{(p),t+1}$ we evaluate daily $VaR_{\theta,t+1/t}(z_{(p),t+1})$. Following the interval forecast method proposed by Christoffersen (1998) we define as efficient the sequence of interval forecasts $\left\{VaR_{\theta,t+1/t}(z_{(p),t+1}), +\infty\right\}_{t\in\square}$, where for every t,

$$E_t\left(I_{t+1}\left(z_{(p),t+1}\right)\right) = \theta,$$

where

$$I_{t+1}\left(z_{(p),t+1}\right) = \begin{cases} 1, & if \quad z_{(p),t+1} \geq VaR_{\theta,t+1/t}(z_{(p),t+1}) \\ 0, & if \quad z_{(p),t+1} < VaR_{\theta,t+1/t}(z_{(p),t+1}) \end{cases}.$$

Note that $I_{t+1}\left(z_{(p),t+1}\right)$ points out when the portfolio losses exceed those estimated. As discussed by Christoffersen, 1998, testing $E_t\left(I_{t+1}\left(z_{(p),t+1}\right)\right) = \theta$ for all t is equivalent to test-

ing that the sequence $\left\{I_{t+1}\left(z_{(p),t+1}\right)\right\}_{t\in\square}$ is an i.i.d. sample from a Bernoulli distribution with parameter θ. That is, we can say that a sequence of interval forecast $\left\{VaR_{\theta,t+1/t},+\infty\right\}_{t\in\square}$ has correct conditional coverage if $\left\{I_{t+1}\right\}$ □ Bern (θ), for every time t. Interval forecasts can be evaluated conditionally or unconditionally, that is, with or without reference to the information available at each point in time.

Table 4

This table summarizes conditional and unconditional coverage tests where the null hypothesis is tested against the alternative at a confidence level 95%. We write "a" when VaR estimates are "acceptably accurate" and we write "r" when we reject the null hypothesis.

	Student's *t*-distribution				Gaussian Distribution		STABLE DISTRIBUTIONS					
	v=4, λ=0.97 VaR 1%		Varying v and λ VaR 1%		Varying λ VaR 1%		α=1.7444 p=0.55, q=1, λ=0.97 VaR 1%		Varying α and λ p=0.55, VaR 1%		Asymmetric varying λ α=1.7557 p=0.549 VaR 1%	
	LR_{uc}	LR_{cc}	LR_{uc}	LR_{cc}	LR_{uc}	LR_{cc}	LR_{uc}	LR_{cc}	LR_{uc}	LR_{cc}	LR_{uc}	LR_{cc}
Port.1	r	a	a	a	r	a	a	a	a	a	a	a
Port.2	r	a	a	a	r	a	a	a	a	a	a	a
Port.3	r	a	a	a	r	a	a	a	a	a	a	a
Port.4	a	a	r	a	r	r	a	a	a	a	a	a
Port.5	r	a	a	a	r	a	a	a	a	a	a	a
Port.6	r	a	a	a	r	a	a	a	a	a	a	a
Port.7	r	r	r	a	r	r	a	a	a	a	a	a
Port.8	r	a	a	a	r	a	a	a	a	a	a	a
Port.9	a	a	a	a	r	r	a	a	a	a	a	a
Port.10	a	a	r	a	r	r	a	a	a	a	a	a
Port.11	r	r	a	a	r	a	a	a	a	a	a	a
Port.12	r	a	a	a	r	a	a	a	a	a	a	a
Port.13	a	a	a	a	r	a	a	a	a	a	a	a
Port.14	r	a	a	a	r	r	a	a	a	a	a	a
Port.15	r	a	a	a	r	a	a	a	a	a	a	a
Port.16	a	a	r	a	r	r	a	a	a	a	a	a
Port.17	r	a	a	a	a	a	a	a	a	a	a	a
Port.18	r	a	a	a	r	a	a	a	a	a	a	a
Port.19	r	a	r	a	r	a	a	a	a	a	a	a
Port.20	r	r	a	a	r	a	r	a	r	a	a	a
Port.21	r	a	a	a	r	a	a	a	a	a	a	a
Port.22	a	a	r	a	r	r	a	a	a	a	a	a
Port.23	a	a	r	r	r	r	a	a	a	a	a	a
Port.24	r	a	r	a	r	r	a	a	a	a	a	a
Port.25	a	a	r	a	r	r	a	a	a	a	a	a

Unconditional coverage test. In order to test the unconditional coverage hypothesis, the null hypothesis $E\left(I_{t+1}\right)=\theta$ should be tested against the alternative $E\left(I_{t+1}\right)\neq\theta$, given inde-

pendence. It can be done using the appropriate likelihood ratio statistic. The likelihood, under the null hypothesis, is $L\left(\theta; I_1,...,I_n\right)=\left(1-\theta\right)^{n_0}\ \theta^{n_1}$ and, under the alternative, is $L\left(q; I_1,...,I_n\right)=\left(1-q\right)^{n_0}\ q^{n_1}$, where n_1 is the number of times for which

$$z_{(p),t+1} \geq VaR_{\theta,t+1/t}\left(z_{(p),t+1}\right),$$

n_0 is the number of times for which

$$z_{(p),t+1} < VaR_{\theta,t+1/t}\left(z_{(p),t+1}\right).$$

Table 5

This table summarizes conditional and unconditional coverage tests where the null hypothesis is tested against the alternative at a confidence level 95%. We write "a" when VaR estimates are "acceptably accurate" and we write "r" when we reject the null hypothesis.

	Student's *t*-distribution				Gaussian Distribution		STABLE DISTRIBUTIONS					
	v=4, λ=0.97 VaR 5%		Varying *v* and λ VaR 5%		Varying λ VaR 5%		α=1.7444 p=0.55, λ=0.97 VaR 5%		Varying α and λ p=0.55, VaR 5%		Asymmetric varying λ α=1.7557 p=0.549 VaR 5%	
	LR_{uc}	LR_{cc}	LR_{uc}	LR_{cc}	LR_{uc}	LR_{cc}	LR_{uc}	LR_{cc}	LR_{uc}	LR_{cc}	LR_{uc}	LR_{cc}
Port.1	r	a	a	a	a	a	a	a	a	a	a	a
Port.2	r	r	a	a	a	a	a	a	a	a	a	a
Port.3	r	r	a	a	a	a	r	a	r	a	a	a
Port.4	r	a	a	a	r	a	r	a	a	a	r	a
Port.5	r	a	a	a	a	a	a	a	a	a	a	a
Port.6	r	a	a	a	a	a	a	a	a	a	a	a
Port.7	a	a	a	a	r	a	a	a	a	a	a	a
Port.8	r	a	a	a	r	a	r	a	a	a	r	a
Port.9	r	a	a	a	r	a	r	a	a	a	a	a
Port.10	a	a	a	a	r	r	a	a	a	a	r	a
Port.11	r	r	a	a	a	a	a	a	a	a	a	a
Port.12	r	r	a	a	a	a	r	a	a	a	a	a
Port.13	r	a	a	a	r	a	a	a	a	a	a	a
Port.14	r	a	a	a	a	a	a	a	a	a	a	a
Port.15	r	a	a	a	a	a	a	a	a	a	a	a
Port.16	r	a	a	a	r	a	r	a	a	a	a	a
Port.17	r	a	a	a	r	a	r	a	r	a	a	a
Port.18	r	a	a	a	a	a	a	a	a	a	a	a
Port.19	r	a	a	a	a	a	a	a	a	a	a	a
Port.20	r	r	a	a	a	a	r	a	a	a	r	a
Port.21	r	a	a	a	a	a	a	a	a	a	a	a
Port.22	r	a	a	a	a	a	r	a	r	a	a	a
Port.23	a	a	a	a	r	a	a	a	a	a	r	a
Port.24	r	a	a	a	r	a	r	a	r	a	a	a
Port.25	r	a	a	a	a	a	a	a	a	a	a	a

Thus, a standard likelihood ratio test for unconditional coverage is given by:

$$LR_{uc} = -2\log\left[\frac{L(\theta; I_1,...,I_n)}{L(\tilde{q}; I_1,...,I_n)}\right] = 2\left[\log\left((1-\tilde{q})^{n_0}\,\tilde{q}^{n_1}\right) - \log\left((1-\theta)^{n_0}\,\theta^{n_1}\right)\right],$$

where $\tilde{q} = \frac{n_1}{n_1 + n_0}$ is the maximum likelihood estimate of *q*. Note that LR_{uc} has an asymptotic $\chi^2(1)$ distribution. The LR_{uc} test is an unconditional test since it counts the times that $z_{(p),t+1} < VaR_{\theta,t+1/t}(z_{(p),t+1})$ over the entire period. However, it is important to consider the conditional accuracy of interval forecasts when the variance changes over the time. As a matter of fact, interval forecasts that ignore variance dynamics could have incorrect conditional coverage at any given time and may have correct unconditional coverage.

Conditional coverage test. Since efficient VaR estimates exhibit the property of correct conditional coverage, then the sequence $\left\{I_{t+1}\left(z_{(p),t+1}\right)\right\}_{t\in\square}$ must exhibit both correct unconditional coverage and serial independence. Therefore, in order to test the conditional coverage hypothesis, Christoffersen (1998) proposed a likelihood ratio statistic LR_{cc} that is a joint test of these two properties, i.e. $LR_{cc} = LR_{uc} + LR_{ind}$ and which is asymptotically distributed $\chi^2(2)$. The LR_{ind} statistic is the likelihood ratio statistic for the null hypothesis of serial independence against an explicit first-order Markov alternative. Considering that the likelihood under the binary first-order Markov alternative is approximated by $L(\Pi_1; I_1,...,I_n) = (1-\pi_{01})^{n_{00}}\,\pi_{01}{}^{n_{01}}\,(1-\pi_{11})^{n_{10}}\,\pi_{11}{}^{n_{11}}$ where $\pi_{ij} = \Pr(I_{t+1} = j / I_t = i)$ and n_{ij} is the number of observations with value *i* followed by *j*, then the likelihood ratio test for conditional coverage is given by:

$$LR_{cc} = -2\log\left[\frac{L(\theta; I_1,...,I_n)}{L(\tilde{\Pi}_1; I_1,...,I_n)}\right] =$$

$$= 2\left[\log\left((1-\tilde{\pi}_{01})^{n_{00}}\,\tilde{\pi}_{01}{}^{n_{01}}\,(1-\tilde{\pi}_{11})^{n_{10}}\,\tilde{\pi}_{11}{}^{n_{11}}\right) - \log\left((1-\theta)^{n_0}\,\theta^{n_1}\right)\right] \square\ \chi^2(2),$$

where $\tilde{\pi}_{ij}$ is the maximum likelihood estimate of π_{ij}, which is simply ratios of the count of the appropriate cells, i.e. $\tilde{\pi}_{01} = \frac{n_{01}}{n_{00} + n_{01}}$ and $\tilde{\pi}_{11} = \frac{n_{11}}{n_{10} + n_{11}}$.

Tables 4 and 5 report the results of conditional and unconditional coverage tests. In particular, in Table 4 we write the results for each 1% percentiles VaR estimates and in Table 5 we test 5% percentiles VaR estimates. In both tables, we test the null hypothesis against the alternative at a 95% level of confidence. We write "*a*" when VaR estimates are "acceptably accurate" and we write "*r*" when they are "inaccurate" and we reject the null hypothesis. These tests partially confirm the previous basic backtest analysis. In particular, we observe that generally Gaussian and Student's *t*-copula with fixed degrees of freedom do not offer adequate performance, while the stable models and the Student's *t*-copula with variable degrees of freedom present better performance. The asymmetric stable model is the model that presents better performance among all the other models.

4. A backtesting analysis of the time rules with simulated and real data

It is well known that the classical Gaussian time rule cannot be applied to Riskmetrics EWMA model. However, practitioners continue to use it in order to evaluate the exposure to risk of their portfolio position. In this section, we value the misspecification of this incorrect use for the Elliptical and the stable model. In particular, we first value the effect of time rules using the same previous data and then we value its impact considering simulating data.

4.1. Time rules backtesting with historical data

In order to assess the reliability of the time rules proposed to compute VaR for the auto-regressive models above, we consider the three previous backtest methods for evaluating the Value at Risk estimates of 25 random portfolios. During the period of 15/11/93-30/01/98 we have examined daily, 10-day, and 60-day returns following: Gaussian distribution (varying λ), Student's t distributions (varying λ and v), Stable sub-Gaussian distribution (with p=0.55, α=1.7444, λ=0.97) and stable asymmetric distribution (with β_γ=-1, α=α_γ=1.7557 and p=0.549). We compute the interval forecasts of the above 25 portfolios over a period of 711 days, using the time aggregation rules (4), (8), and considering θ=99% and θ=95%.

Table 6

This table summarizes the average of backtesting results of EWMA with Gaussian and Student's t distributions, SEWMA and stable asymmetric models for ϑ=95% and ϑ=99%. In the second line we summarize conditional and unconditional coverage tests where the null hypothesis is tested against the alternative at a confidence level 95%. In particular, we forecast VaR estimates for 10 days returns and 60 days returns, using the respective time rules. Then we determine the average of percentages of times the returns fall outside the confidence interval. Finally, we determine the percentages of times VaR estimates are "acceptably accurate".

	Student's t distribution	Gaussian Distribution	STABLE DISTRIBUTIONS	
	Varying v and λ	Varying λ	Stable EWMA α=1.7444 p=0.55, λ=0.97	Asymmetric varying λ α=1.7557 p=0.549
10 days VaR 5%	0.050889 LR_{uc} 72%; LR_{cc} 96%	0.051284 LR_{uc} 96%; LR_{cc} 96%	0.038050 LR_{uc} 68%; LR_{cc} 96%	0.032954 LR_{uc} 48%; LR_{cc} 96%
10 days VaR 1%	0.017673 LR_{uc} 40%; LR_{cc} 68%	0.018291 LR_{uc} 76%; LR_{cc} 68%	0.005675 LR_{uc} 60%; LR_{cc} 88%	0.001814 LR_{uc} 12%, LR_{cc} 76%
60 days VaR 5%	0.024329 LR_{uc} 84%; LR_{cc} 36%	0.026434 LR_{uc} 88%; LR_{cc} 44%	0.015233 LR_{uc} 12%; LR_{cc} 32%	0.012590 LR_{uc} 4%; LR_{cc} 20%
60 days VaR 1%	0.008420 LR_{uc} 48%; LR_{cc} 36%	0.009027 LR_{uc} 48%, LR_{cc} 28%	0.002024 LR_{uc} 8%; LR_{cc} 16%	0.000000 LR_{uc} 0%; LR_{cc} 0%

The first basic empirical analysis compares the results obtained from the backtest carried out among the elliptical EWMA models and the stable asymmetric model for θ=99% and θ=95%. In view of this comparison, we assume the same parameters of the previously analyzed models. Then, we apply the different time rules (4), (8) in order to forecast VaR estimates and compare their performance. Table 6 summarizes the average of backtesting results using stable and elliptical EWMA models. That is, for each portfolio we determine the percentages of times the profits/losses fall outside the confidence interval; then we take the average of these percentages among the portfolios taken into account. In particular, when we consider returns on a period of 10 days the stable models and their time rules overestimate the losses, while the Student's t and the Gaussian time rules underestimate the losses. With 60-day returns all the time rules overestimate the losses, instead. These results are substantially confirmed when we use the conditional and unconditional coverage tests. Table 6 reports the results in average of these statistics. In particular, for every portfolio, we first test if VaR estimates are "acceptably accurate" or they are "inaccurate".

Then, we compute the percentage of portfolios which are acceptably accurate. As we can see from Table 6, when we use time aggregation rules for 10-day returns the stable EWMA model presents the best performance for θ=99%, while the Student's *t* model and the classic Gaussian EWMA model perform better for θ=95%. When we consider 60-day returns, all the models do not perform well, even if the Gaussian and Student's *t* times rules perform better than the stable Paretian ones.

4.2. Time rules backtesting with simulated data

One of the main problems of using historical data consists in the fact that we cannot value the impact of the previous time rules when the data follow exactly the autoregressive model proposed. For this reason, in order to value the misspecification of these aggregation rules we have also used simulated data. In particular, we simulate the behavior of the previous models and we value these time rules on simulated data. The main problem to generate EWMA-type scenarios is due to the non-stationariness of the process that implies that the variance process converges a.s. to zero (see Nelson, 1990). This simple observation implicitly suggests that time rules are not valid. In this comparison, we simulate 5000 scenarios with Gaussian, *t*-Student (with 7 degrees of freedom), α-stable sub-Gaussian (with α=1.25 and p=0.55) EWMA models and stable asymmetric model (with α=1.75, β_Y=-1, p=0.55, with fixed $\sigma_Y = 0.00312$ and fixed vector b_t). Then we compare the performance of time rules applied to the different models. The Student's *t* scenarios are generated using formula (13). In order to simulate stable distributions, we follow Chambers, Mallows and Stuck's algorithm (see Chambers et al., 1976). Moreover, in order to generate an α-Stable sub-Gaussian vector Z with null mean and dispersion matrix Q, we can generate a Gaussian vector G with variance covariance matrix Q and then we multiply it for the square root of a stable subordinator $X \sim S_{\alpha/2}\left(2\left(\cos\left(\frac{\pi\alpha}{4}\right)\right)^{2/\alpha}, 1, 0\right)$ independent from the vector G, i.e. $Z \sim \sqrt{X}G$. In particular, we consider for all models the decay factor $\lambda = 0.98$. Even in this case, we compute the interval forecasts of the above 25 portfolios over the 5000 scenarios, using the time aggregation rules (4), (8), and considering θ=99% and θ=95%.

Table 7

This table summarizes the average of backtesting results of EWMA with Gaussian and Student's *t* distributions, stable EWMA and stable asymmetric models for ϑ=95% and ϑ=99% based on simulated data. In the second line we summarize conditional and unconditional coverage tests where the null hypothesis is tested against the alternative at a confidence level 95%. In particular, we forecast VaR estimates for 10; 20; 30; 60 and 120 days returns, using the respective time rules on the simulated EWMA and asymmetric models. Then we determine the average of percentages of times the returns fall outside the confidence interval. Finally, we determine the percentages of times VaR estimates are "acceptably accurate".

	Student's *t* distribution	Gaussian Distribution	STABLE DISTRIBUTIONS	
	v=7 and λ=0.98	λ=0.98	SEWMA α=1.25 p=0.55, λ=0.98	Asymmetric stable α=1.75 p=0.55, λ=0.98
10 days VaR 5%	0.0515 LR_{uc} 100%; LR_{cc} 100%	0.0521 LR_{uc} 100%; LR_{cc} 100%	0.0496 LR_{uc} 100%; LR_{cc} 100%	0.05003 LR_{uc} 100%; LR_{cc} 100%
10 days VaR 1%	0.0117 LR_{uc} 96%; LR_{cc} 96%	0.0121 LR_{uc} 96%; LR_{cc} 96%	0.00989 LR_{uc} 96%; LR_{cc} 100%	0.01048 LR_{uc} 96%; LR_{cc} 100%
20 days VaR 5%	0.0486 LR_{uc} 96%%; LR_{cc} 100%	0.0495 LR_{uc} 100%; LR_{cc} 100%	0.04944 LR_{uc} 100%; LR_{cc} 100%	0.04912 LR_{uc} 100%; LR_{cc} 100%
20 days VaR 1%	0.01087 LR_{uc} 92%; LR_{cc} 96%	0.0115 LR_{uc} 92%; LR_{cc} 92%	0.00967 LR_{uc} 92%; LR_{cc} 96%	0.00981 LR_{uc} 96%; LR_{cc} 100%

Table 7 (continuous)

30 days VaR 5%	0.03891 LR_{uc} 80%; LR_{cc} 92%	0.04128 LR_{uc} 80%; LR_{cc} 96%	0.04565 LR_{uc} 88%; LR_{cc} 96%	0.04645 LR_{uc} 92%; LR_{cc} 96%
30 days VaR 1%	0.00772 LR_{uc} 80%; LR_{cc} 92%	0.00826 LR_{uc} 84%; LR_{cc} 96%	0.00767 LR_{uc} 84%; LR_{cc} 96%	0.00841 LR_{uc} 88%, LR_{cc} 96%
60 days VaR 5%	0.02635 LR_{uc} 72%; LR_{cc} 84%	0.02343 LR_{uc} 68%; LR_{cc} 80%	0.03164 LR_{uc} 72%; LR_{cc} 84%	0.02964 LR_{uc} 72%; LR_{cc} 84%
60 days VaR 1%	0.00536 LR_{uc} 52%; LR_{cc} 60%	0.00644 LR_{uc} 52%; LR_{cc} 60%	0.00623 LR_{uc} 56%; LR_{cc} 68%	0.00694 LR_{uc} 56%; LR_{cc} 68%
120 days VaR 5%	0.02854 LR_{uc} 44%; LR_{cc} 56%	0.02941 LR_{uc} 48%; LR_{cc} 56%	0.03421 LR_{uc} 52%; LR_{cc} 64%	0.02817 LR_{uc} 48%; LR_{cc} 60%
120 days VaR 1%	0.004235 LR_{uc} 32%; LR_{cc} 44%	0.005026 LR_{uc} 36%; LR_{cc} 52%	0.006041 LR_{uc} 40%; LR_{cc} 56%	0.004964 LR_{uc} 36%; LR_{cc} 56%

Table 7 reports the comparison among the different models when we use simulated data. As we should expect, we obtain a better performance of time rules using simulated data, in particular, when we consider a temporal horizon of 10 days. The proposed empirical results show the best performance of the simulated stable models as compared of to the Gaussian and the Student's *t* one (see Table 7). On the other hand, this simulation analysis underlines that scaling up volatility forecasts may sometimes lead to results that do not make much sense when estimates of 1-day volatilities are used to predict long time volatilities. Thus, as confirmed by the theory (see Diebold et al., 1998; Lamantia et al., 2006), the time rules (4), (8) can be limitedly used to forecast future losses if returns follow the above models. In particular, when we consider a temporal aggregation of 60-120 days, the effect of the error in the approximation is big enough as it appears by the empirical misspecification.

5. Conclusions

This paper proposes and compares alternative models for the VaR calculation. In the first part we describe and recall several elliptical and stable Paretian parametric models. The empirical comparison confirms that when the percentiles are below 5%, the hypothesis of normality of the conditional return distribution determines intervals of confidence whose forecast ability is low. As a matter of fact, the return distributions are asymmetric and leptokurtic and the hypothesis of normality is usually rejected under statistical test. The stable Paretian models and the Student's *t*-copula have shown very good performance in predicting future losses. Among the alternative models proposed, the α-stable densities are reliable in the VaR calculation and are characterized by an approximating temporal aggregation rule. Moreover, some stable parametric models present better performance for smaller percentiles and they reveal a high degree of efficiency for large portfolios. Thus it is reasonable to believe that implementing stable models for online VaR calculation is a realistic issue. Finally, the performance analysis of the proposed time aggregation rules has shown an adequate capacity of all models to predict future losses when we assume a temporal horizon of 10 days. In particular, we observe a better performance of the stable Paretian time rules when we use simulated data. However, when we consider 60-120 day returns all the temporal aggregation models do not present very good results when they are based on historical and simulated data. This empirical comparison confirms that when the temporal horizon is too large the time rules cannot be applied.

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