

Wolfgang Benner (Germany), Lyudmil Zypkov (Germany)
**A multifactoral Cross-Currency LIBOR Market Model
 with an FX volatility skew**

Abstract

Based on LIBOR Market Models, we develop a rigorous pricing framework for cross-currency exotic interest rate instruments under a uniform probability measure and in a multifactoral environment that accounts for the empirically observed foreign exchange skew. The model resorts to a stochastic volatility approach with volatility dynamics following a square-root process and is designed to be flexible enough to allow for the incorporation of as much market information as possible. Using the Fourier transform, we produce closed-form valuation formulas for FX options by obtaining an explicit expression for the characteristic function, though in a mildly approximate fashion for the sake of analytical tractability. The main focus is placed on FX markets, in terms of which the calibration of model parameters can be performed on a wide range of FX options expiries and strikes.

Keywords: Cross-Currency LIBOR Market Model, stochastic volatility, Fourier transform, foreign exchange skew, forward probability measure.

JEL Classification: G13, E43, F31.

Introduction

The origins of the proposed Cross-Currency LIBOR Market Model (CCLMM) can be traced back to the need of developing a unified pricing framework for a number of cross-currency exotics. Initially confronted with a hybrid structure that required the simultaneous description of highly correlated interest rate markets, foreign exchange (forex) rate and hazard rate dynamics, the present work gained impetus from the necessity to determine the value of a cross-currency swap, which was to serve as an underlying of various derivative products, at an arbitrary future date. Typically, FX options exhibit a significant volatility skew that manifests itself in the at-the-money (ATM) implied volatility's underestimation of in-the-money (ITM) option prices and overestimation of out-of-the-money (OTM) ones, whereby the ATM implied volatility has been obtained by inversion of an ATM option pricing formula based on a lognormal stochastic evolution of the forward forex rate. Moreover, it seems impossible for the most cross-currency derivatives to choose a particular strike, or a specific maturity of an FX option since they usually represent long-dated exotic structures that either cannot be decomposed into plain-vanilla FX options, or at best depend on FX options for a wide range of strikes and maturities. Aggravating matters even further, exotic cross-currency interest rate derivatives are rarely structured to depend on ATM volatilities. They are usually designed with strikes far away from at-the-money. Hence, the volatility function needs to be calibrated to prices of FX options across all available maturities and strikes as suggested by Piterbarg (2006). He asserts that a model similar to that of Schloegl (2002) based on LIBOR Market Models, yet accounting for forex smiles in a proper manner, and a good FX option calibration algorithm still awaits development. For this purpose, it appears

natural to resort to an extension of the lognormal-type dynamics of the forward forex rate that is based on stochastic volatility.

This paper proposes an integrated CCLMM under a uniform pricing measure in a multifactoral environment that allows for as much flexibility as possible in calibrating model parameters to market data. The pricing measure will be uniform as it will be applicable to (i) simple financial instruments that are affected only by the domestic interest rate market or the foreign interest rate market but not both, as well as to (ii) complex financial instruments that are affected by both the domestic and foreign interest rate markets linked by the forex market. With the intention to derive valuation formulas, we deflate all stochastic price processes using a single numeraire regardless of the market the price process belongs to or is affected by, thus ensuring pricing consistency between the markets and allowing the evaluation of complex financial structures within a LIBOR Market Model setup. The model design must be capable of reflecting market implied volatilities and exogenously assigned correlation structures between the interest rates and FX dynamics. However, the main focus will be placed on the calibration to FX options for various maturities and strikes simultaneously, while retaining one-factor assumptions for both interest rate markets. Though somewhat restrictive at first glance, this choice keeps the number of model parameters to be calibrated low affecting high speed of calibration without sacrificing accuracy of valuation. In addition, the model developed here can easily be used as a stepping stone to incorporating interest rate volatility smiles on a multi-currency basis, which remains a subject of future research. The various extensions of the forward LIBOR models could serve as a starting point of this effort. One possibility would be the postulation of alternate interest rate dynamics such as local volatil-

ity type of extensions based on constant elasticity of variance (CEV) processes pioneered by Andersen et al. (2000), or the adoption of a displaced-diffusion approach as elaborated, for example, by Benner et al. (2007). Jump-diffusions are treated in Glasserman et al. (2003a, 2003b), but have not gained much acceptance due to their producing of non-time-homogeneous volatility term structures and some other calibration complications. Finally as the modelling technique with probably the greatest explanatory power, the inclusion of stochastic volatility in the LMM is considered by three main research streams: Andersen et al. (2005) and Andersen et al. (2002), on which Piterbarg (2003) builds using the method of calibration by parameter averaging as described in Piterbarg (2005a, 2005b) and providing formulas that relate market and model skews and volatilities directly without the need to develop closed-form solutions of European option valuation problems. Joshi et al. (2003) choose a distinctly different way of analyzing the evolution of the swaption volatility matrix over time by assuming a specific time-homogeneous instantaneous volatility function whose parameters are allowed to vary stochastically.

The paper is organized as follows. Section 1 develops a unified pricing framework under a uniform domestic forward measure. It determines both the dynamics of the domestic/foreign LIBORs and the forward forex rate with stochastic volatility. The reason why forward forex rates are being modelled directly is that, by definition, they represent price processes of tradable securities as opposed to spot forex rates. In fact, each forward forex rate follows a martingale under its natural forward measure, so that its dynamics are, under such a measure, fully specified by its volatility process. Section 2 derives an FX option pricing formula with stochastic forex volatilities based on mildly approximate assumptions in order to preserve the analytical tractability of the model. It also offers an elaborate overview of the implemented calibration procedure. The last section concludes with a brief summary of the main results and some suggestions for future work. For the sake of lucid presentation, all purely technical details are reserved to the appendices at the end of the paper.

1. Cross-Currency LMM under uniform probability

$$L_{t_i}(t) = \alpha_i^{-1} \left(\frac{B(t, t_i)}{B(t, t_{i+1})} - 1 \right), \quad L_{t_i}^f(t) = \alpha_i^{-1} \left(\frac{B^f(t, t_i)}{B^f(t, t_{i+1})} - 1 \right) \quad \text{with } \alpha_i = t_{i+1} - t_i = 1.$$

In particular:

$$L(t, t, t_1) = \alpha_0^{-1} \left(\frac{1}{B(t, t_1)} - 1 \right), \quad L^f(t, t, t_1) = \alpha_0^{-1} \left(\frac{1}{B^f(t, t_1)} - 1 \right) \quad \text{with } \alpha_0 = t_1 - t \leq 1.$$

measure

1.1. Definitions. Given a filtered probability space

$$\left(\Omega, \{F_t\}_{t \in [t_0, t_N]}, P^{t_N} \right)$$

satisfying the usual conditions, let the tuple $\{W_{t_N}^{(d)}(t)\}_{t \in [t_0, t_N]}$ ^{d=1,2,3,4} denote a d-dimensional Brownian motion that introduces the source of uncertainty to the correlated dynamics of the foreign exchange market, both the domestic and foreign interest rate markets and to the mean-reverting square root process of the common volatility $V(t)$ shared equally by forward forex rates of any maturity. In addition, we assume that the filtration $\{F_t\}_{t \in [t_0, t_N]}$ is the usual P^{t_N} -augmentation of the natural filtration generated by $\{W_{t_N}^{(d)}(t)\}_{t \in [t_0, t_N]}$ ^{d=1,2,3}, so that it is right-continuous and complete. A common set of LIBOR maturities $0 = t_0 < t_1 < \dots < t_N$ is defined for both the domestic and the foreign currency markets. May the following symbols $L(s, t, T)$ and $L^f(s, t, T)$ indicate domestic and foreign (forward) LIBORs as of time s , starting at time t and maturing at time T respectively with $s \leq t \leq T$. For the sake of simplicity, $L_{t_i}(t) = L(t, t_i, t_{i+1})$ and $L_{t_i}^f(t) = L^f(t, t_i, t_{i+1})$ are designated to represent one-year forward rates starting at t_i and ending at $t_{i+1} = t_i + 1$ with $i = 0, \dots, N - 1$. Moreover, $L(t, t, t_i)$ and $L^f(t, t, t_i)$ specify spot rates, whereas $Q(t)$ stands for the spot forex rate as units of domestic currency per unit of foreign currency. The forward forex rate, at which investors can buy or sell foreign currency for settlement at a future date, is determined by $FX_i(t) = B^f(t, t_i)Q(t)/B(t, t_i)$.

In terms of zero-coupon bonds, $B(t, t_i)$ and $B^f(t, t_i)$ denote the domestic and foreign bond respectively, while the LIBORs are given by:

Their stochastic evolution is characterized solely by the respective volatility functions $\gamma_i(t)$ and $\gamma_i^f(t)$, which are assumed deterministic within the main framework and can be calibrated independently for the domestic and foreign LIBORs to single-currency caps and swaptions using the well-known techniques for the single-currency LMM suggested by Rebonato (2002) and Brigo et al. (2002). To capture the implied volatility's functional dependence on the corresponding interest rate option's strike, a displaced-diffusion approach according to Rubinstein (1983) can be adopted, as shown by Benner et al. (2007). However, it will not be further pursued in

this place since we primarily concentrate on retaining sufficient control over forex smiles and developing a practicable FX option calibration algorithm for a wide range of maturities and across a variety of strikes. For convenience, the terminal forward probability measure P_{t_N} associated with the domestic bond maturing at the terminal date t_N is chosen to be the uniform martingale measure throughout the paper. Though of marginal importance to the analysis, the structure of both bond volatilities $\sigma^{(f)}(t, t_N)$ is nonetheless needed and is determined according to Benner et al. (2007):

$$\sigma^{(f)}(t, t_N) = -\sum_{i=1}^{N-1} \frac{\alpha_i L^{(f)}(t, t_i, t_{i+1})}{1 + \alpha_i L^{(f)}(t, t_i, t_{i+1})} \gamma^{(f)}(t, t_i, t_{i+1}) - \frac{\alpha_0 L^{(f)}(t, t, t_1)}{1 + \alpha_0 L^{(f)}(t, t, t_1)} \gamma^{(f)}(t, t, t_1).$$

1.2. Modeling forward forex rates with stochastic volatility. We begin by assuming the following general dynamics for the CCLMM:

$$\begin{aligned} \frac{dL_i(t)}{L_i(t)} &= \mu_i(t)dt + \gamma_i(t)dW_1^{P_{t_N}}(t), & \frac{dL_i^f(t)}{L_i^f(t)} &= \mu_i^f(t)dt + \gamma_i^f(t)dW_2^{P_{t_N}}(t), \\ \frac{dQ(t)}{Q(t)} &= \mu^q(t)dt + \sigma^q(t)dW_Q^{P_{t_N}}(t), & W_{1,2,Q}^{P_{t_N}} &\rightarrow P_{t_N} - \text{BROWNIAN motions.} \\ dW_1(t)dW_2(t) &= \rho_{12}dt, & dW_1(t)dW_Q(t) &= \rho_{1Q}dt, & dW_2(t)dW_Q(t) &= \rho_{2Q}dt \end{aligned} \tag{1}$$

The last forward forex rate within the exemplary tenor structure represents a martingale under P_{t_N} :

$$\begin{aligned} \frac{dFX_N(t)}{FX_N(t)} &= \sigma^f(t, t_N)dW_2^{P_{t_N}}(t) + \sigma^q(t)dW_Q^{P_{t_N}}(t) - \sigma(t, t_N)dW_1^{P_{t_N}}(t) = \sigma_N^{fx}(t)dW_3^{P_{t_N}}(t), \text{ where} \\ \sigma_N^{fx}(t)^2 &= \sigma_N^f(t)^2 + \sigma^q(t)^2 + \sigma_N(t)^2 - 2\sigma_N^f(t)\sigma_N(t)\rho_{1,2} - 2\sigma^q(t)\sigma_N(t)\rho_{1,Q} + 2\sigma^q(t)\sigma_N^f(t)\rho_{2,Q} \\ dW_1(t)dW_3(t) &= \rho_{13}dt, & dW_2(t)dW_3(t) &= \rho_{23}dt \end{aligned} \tag{2}$$

Strictly speaking, $\sigma_N^{fx}(t)$ is a stochastic quantity through its dependence on the realization of the LIBOR rates³. Though, it could be made conditionally deterministic to a high degree of accuracy by some quite sophisticated and extremely precise approximation methods, or simply by classical “drift-freezing” techniques⁴. Yet another approach of directly calibrating $\sigma_N^{fx}(t)$ as a (deterministically) variable model parameter is being pursued henceforth since the forward forex rate represents the price process of a tradable asset denominated by the corresponding numeraire, hence an observable security, as it has already been mentioned above. The decomposition of the volatility in (2) serves solely for the purpose of enabling us to determine the drift of any forward forex rate prior to the terminal one

by switching from the natural to the terminal measure, as will be shown shortly.

Such a model of the forward forex rate can safely be used to price FX options with different maturities but the same strike. When simultaneously pricing options with various strikes, however, the natural question arises as to how to account for the usually observed smile effect. For this reason, we resort to an extension of the lognormal-type dynamics of the forward forex rate beyond the geometric BROWNIAN motion and postulate a stochastic volatility evolution in conformity with Heston (1993) based on a common volatility $V(t)$ that follows a mean-reverting square-root process under the physical probability measure:

$$\frac{dFX_N(t)}{FX_N(t)} = \sigma_N^{fx}(t)\sqrt{V(t)}dW_3^{P_N}(t), \quad dV(t) = \alpha(\theta - V(t))dt + \xi\sqrt{V(t)}dW_4(t) \tag{3}$$

$$dW_1(t)dW_4(t) = \rho_{14}dt, \quad dW_2(t)dW_4(t) = \rho_{24}dt, \quad dW_3(t)dW_4(t) = \rho_{34}dt$$

Any previous forward forex rate no longer follows a martingale, but its dynamics under P_N can be determined according to:

$$\text{With } FX_i(t) = \frac{B^f(t, t_i)Q(t)}{B(t, t_N)} \bigg/ \frac{B(t, t_i)}{B(t, t_N)} \Rightarrow \frac{dFX_i(t)}{FX_i(t)} = \mu_i^{fx}(t)dt + \sigma_i^{fx}(t)\sqrt{V(t)}dW_3^{P_N}(t),$$

$$\text{where } \mu_i^{fx}(t)dt = -\left(\sigma_i(t)dW_1^{P_N}(t) - \sigma_N(t)dW_1^{P_N}(t)\right)\square$$

$$\left(\sigma_i^f(t)dW_2^{P_N}(t) + \sigma^q(t)dW_Q^{P_N}(t) - \sigma_N(t)dW_1^{P_N}(t) - \sigma_i(t)dW_1^{P_N}(t) + \sigma_N(t)dW_1^{P_N}(t)\right)$$

$$= -\sigma_i^{fx}(t)\sqrt{V(t)}dW_3^{P_N}(t) \sum_{j=i}^{N-1} \frac{\alpha_j L(t, t_j, t_{j+1})}{1 + \alpha_j L(t, t_j, t_{j+1})} \gamma_{t_j}(t) dW_1^{P_N}(t) \Rightarrow$$

$$\mu_i^{fx}(t) = -\sum_{j=i}^{N-1} \frac{\alpha_j L(t, t_j, t_{j+1})}{1 + \alpha_j L(t, t_j, t_{j+1})} \gamma_{t_j}(t) \sigma_i^{fx}(t) \sqrt{V(t)} \rho_{13},$$

while the volatility process evolves under the uniform martingale measure as follows:

$$\text{With } dW_4(t) = dW_4^{P_N}(t) + \frac{dB(t, t_N)}{B(t, t_N)} dW_4(t) = dW_4^{P_N}(t) + \sigma(t, t_N) \rho_{14} dt \Rightarrow$$

$$dV(t) = \left[\alpha(\theta - V(t)) + \xi\sqrt{V(t)}\sigma(t, t_N)\rho_{14} \right] dt + \xi\sqrt{V(t)}dW_4^{P_N}(t).$$

Finally, the drift functions of both term structures of interest rates under P_N remain to be computed. It is well-understood that the drift of the domestic LIBOR takes on the expression:

$$\mu_i(t) = -\sum_{j=i+1}^{N-1} \frac{\alpha_j L(t, t_j, t_{j+1}) \gamma_{t_j}(t) \gamma_{t_i}(t)}{1 + \alpha_j L(t, t_j, t_{j+1})}, \quad i < N-1.$$

A sequential procedure starting with the terminal foreign LIBOR, and moving backwards until the spot LIBOR rate is reached, renders the evolution of

$$d \frac{L_{t_{N-1}}^f(t) B^f(t, t_N) Q(t)}{B(t, t_N)} = \left[\mu_{N-1}^f(t) + \gamma_{N-1}^f(t) \sigma_N^{fx}(t) \sqrt{V(t)} \rho_{23} \right] \frac{L_{t_{N-1}}^f(t) B^f(t, t_N) Q(t)}{B(t, t_N)} dt +$$

$$\frac{L_{t_{N-1}}^f(t) B^f(t, t_N) Q(t)}{B(t, t_N)} \gamma_{N-1}^f(t) dW_2^{P_N}(t) + \frac{L_{t_{N-1}}^f(t) B^f(t, t_N) Q(t)}{B(t, t_N)} \sigma_N^{fx}(t) \sqrt{V(t)} dW_3^{P_N}(t) \Rightarrow$$

$$\mu_{N-1}^f(t) = -\gamma_{N-1}^f(t) \sigma_N^{fx}(t) \sqrt{V(t)} \rho_{23}.$$

Applying the same reasoning to an arbitrary foreign LIBOR prior to the terminal one, we obtain:

$$\text{With } \frac{L_i^f(t) B^f(t, t_{i+1}) Q(t)}{B(t, t_N)} = L_i^f(t) FX_{i+1}(t) \frac{B(t, t_{i+1})}{B(t, t_N)} \text{ and } i < N-1 \Rightarrow$$

$$\frac{d \left[\frac{L_i^f(t) B^f(t, t_{i+1}) Q(t)}{B(t, t_N)} \right]}{\frac{L_{t_{N-1}}^f(t) B^f(t, t_N) Q(t)}{B(t, t_N)}} = \left(\mu_i^f + \gamma_{t_i}^f(t) \sigma_{i+1}^{fx}(t) \sqrt{V(t)} \rho_{23} + \gamma_{t_i}^f(t) \sum_{j=i+1}^{N-1} \frac{\alpha_j L_{t_j}(t) \gamma_{t_j}(t)}{1 + \alpha_j L_{t_j}(t)} \rho_{12} \right) dt$$

$$\gamma_{t_i}^f(t) dW_2^{P_N}(t) + \sigma_{i+1}^{fx}(t) \sqrt{V(t)} dW_3^{P_N}(t) + \sum_{j=i+1}^{N-1} \frac{\alpha_j L_{t_j}(t) \gamma_{t_j}(t)}{1 + \alpha_j L_{t_j}(t)} dW_1^{P_N}(t) \Rightarrow$$

$$\mu_i^f = -\gamma_{t_i}^f(t) \sigma_{i+1}^{fx}(t) \sqrt{V(t)} \rho_{23} - \gamma_{t_i}^f(t) \sum_{j=i+1}^{N-1} \frac{\alpha_j L_{t_j}(t) \gamma_{t_j}(t)}{1 + \alpha_j L_{t_j}(t)} \rho_{12}.$$

the foreign term structure at last attainable. Since any traded asset scaled by the respective numeraire will be a martingale under the corresponding measure, we infer that $L_{N-1}^f(t) B^f(t, t_N) Q(t)$, which represents a portfolio of foreign bonds in domestic currency, will be driftless when divided by the numeraire $B(t, t_N)$:

Summarizing the results so far, the proposed Cross-Currency LIBOR Market Model with an incorporated forex smile is fully described by the following system of stochastic differential equations:

$$\begin{aligned} \frac{dL_{t_i}(t)}{L_{t_i}(t)} &= -\gamma_{t_i}(t) \sum_{j=i+1}^{N-1} \frac{\alpha_j L(t, t_j, t_{j+1}) \gamma_{t_j}(t)}{1 + \alpha_j L(t, t_j, t_{j+1})} dt + \gamma_{t_i}(t) dW_1^{P_{t_N}}(t) \\ \frac{dL_{t_i}^f(t)}{L_{t_i}^f(t)} &= -\gamma_{t_i}^f(t) \left(\sigma_{i+1}^{fx}(t) \sqrt{V(t)} \rho_{23} + \sum_{j=i+1}^{N-1} \frac{\alpha_j L_{t_j}(t) \gamma_{t_j}(t)}{1 + \alpha_j L_{t_j}(t)} \rho_{12} \right) dt + \gamma_{t_i}^f(t) dW_2^{P_{t_N}}(t) \\ \frac{dFX_i(t)}{FX_i(t)} &= -\sum_{j=i}^{N-1} \frac{\alpha_j L(t, t_j, t_{j+1}) \gamma_{t_j}(t)}{1 + \alpha_j L(t, t_j, t_{j+1})} \sigma_i^{fx}(t) \sqrt{V(t)} \rho_{13} dt + \sigma_i^{fx}(t) \sqrt{V(t)} dW_3^{P_{t_N}}(t) \\ dV(t) &= \left[\alpha(\theta - V(t)) + \xi \sqrt{V(t)} \sigma(t, t_N) \rho_{14} \right] dt + \xi \sqrt{V(t)} dW_4^{P_{t_N}}(t) \\ dW_m(t) dW_l(t) &= \rho_{ml} dt, \text{ where } m, l = \{1, 2, 3, 4\} \text{ and } m \neq l. \end{aligned} \tag{4}$$

The correlation coefficients ρ_{ml} , $m, l = 1, 2, 3$, can be chosen either by historical estimation, or by parsimonious parameterization of the correlation function¹ and subsequent calibration to the information extracted from occasionally observed prices of quanto interest rate contracts. Assuming that both the domestic and the foreign LIBOR volatilities have previously been calibrated independently to single-currency caps and swaptions as indicated above, the model still needs to be calibrated to available FX options if it is intended to be used as a pricing tool for cross-currency exotics. Therefore, the main purpose of this article, aside from deriving

the CCLMM with a forex smile in (4), consists in the development of an effective and fast calibration algorithm, at the core of which a closed-form FX option valuation formula stands.

2. Option pricing formulas and calibration routines

2.1. FX option valuation by Fourier transform.

Regardless of the model chosen for the evolution of the forward forex rate, the price at t_0 of the i^{th} FX option under the natural measure P_{t_i} and under the equivalent measure P_{t_N} is given by:

$$FXopt_i(t_0) = B(t_0, t_i) E^{P_{t_i}} \left[(FX_i(t) - K)^+ | F_{t_0} \right] = B(t_0, t_N) E^{P_{t_N}} \left[(FX_i(t) - K)^+ \frac{B(t, t_i)}{B(t, t_N)} | F_{t_0} \right].$$

The driftless dynamics of $FX_i(t)$ under their own natural probability measure will produce the correct option price only if the discounting of the payoff is carried out using the appropriate numeraire – in this case the bond maturing at t_i . The use of any other measure will introduce a covariance between the discounting and the payoff itself, for example when the pricing of plain-vanilla options on the whole spectrum of forward forex rates $FX_i(t)$, $i = 1, \dots, N$, is accomplished under a single measure like the terminal one. In order to recover the same option value, this fact has to be compensated for by altering the drift of the forward forex rate as shown previously. However, facing a complex pricing problem, which entails simultaneously several FX rates, a particular measure has to be specified and once it has been chosen, the presence of non-zero drifts is unavoidable and the need to formulate a model for the stochastic evolution of the underlying(s) like (4) is inevitable. This has been the main purpose of our work – the development of a viable pricing model for exotic cross-currency interest rate

instruments, which is at the same time flexible enough to allow for the incorporation of the entire market information, as of FX markets essentially meaning calibration to the whole range of FX options prices across all available maturities and strikes. As a step prior to the actual pricing of derivative structures, the calibration of the model can be carried out under any probability measure since we calibrate to plain-vanilla FX options, whose payoffs involve only a single forward forex rate at a particular point of time. It is acceptable to use a model separately calibrated for each option expiry since vanillas depend on the terminal distribution of the underlying only as opposed to exotics whose values usually depend on the full dynamics through time of a whole range of FX rates. Consequently, we revert every time to the natural measure P_{t_i} , effectively using for calibration always the same model though under different probability measure, to circumvent the unpleasant dependence alluded to above, which will admittedly complicate the producing of a closed-form solution to the option pricing problem.

The value of the FX option can be rewritten in terms of the real part of its Fourier transform, where the only unknown parameter is the conditional charac-

teristic function $\phi_0^i(\cdot)$ of $\ln(FX_i(t)) = Y_i(t)$ $\phi_0^i(u) = E^{P_i} \left(e^{iuY_i(t)} \mid F_{t_0} \right)$, eventually arriving at:

$$\frac{FXopt_i(t_0)}{B(t_0, t_i)} = FX_i(t_0) \left(\frac{1}{2} + \frac{1}{\pi} \int_0^\infty \text{Re} \left[\frac{e^{-iu \ln K} \phi_0^i(u-i)}{iu \phi_0^i(-i)} \right] du \right) - K \left(\frac{1}{2} + \frac{1}{\pi} \int_0^\infty \text{Re} \left[\frac{e^{-iu \ln K} \phi_0^i(u)}{iu} \right] du \right). \tag{5}$$

Similar descriptions of the option price in a different form have already been derived by numerous authors, e.g., Bakshi et al. (2000) and Scott (1997), and numerically determined on the assumption that the characteristic function is known analytically. One disadvantage of the formula above is the singularity of the integrand at the required evaluation point $u = 0$, which ultimately precludes the application of the Fast Fourier Transform (FFT). Therefore, Carr et al. (1999) develop a new analytic expression

for the Fourier transform designed to use the FFT to price options efficiently. In the appendix we propose a different approach, which draws directly upon Lévy's inversion theorem, and avail ourselves of the Gauss-Laguerre Quadrature to obtain the best numerical estimate of the Fourier integrals in (5). Therefore, the option pricing reduces to the calculation of the unknown conditional characteristic function. The dynamics directly relevant to valuing the FX option are:

$$\text{With } \sigma(t, t_i) = - \sum_{j=1}^{i-1} \frac{\alpha_j L_{t_j}(t)}{1 + \alpha_j L_{t_j}(t)} \gamma_{t_j}(t) - \frac{\alpha_0 L_t(t)}{1 + \alpha_0 L_t(t)} \gamma_t(t); dW_m(t) dW_l(t) = \rho_{ml} dt; m, l = \{1, 3, 4\}$$

$$dY_i(t) = -\frac{1}{2} \sigma_i^{fx}(t)^2 V(t) dt + \sigma_i^{fx}(t) \sqrt{V(t)} dW_3^{P_i}(t) \tag{6}$$

$$dV(t) = \left[\alpha(\theta - V(t)) + \xi \sqrt{V(t)} \sigma(t, t_i) \rho_{14} \right] dt + \xi \sqrt{V(t)} dW_4^{P_i}(t)$$

$$\frac{dL_{t_j}(t)}{L_{t_j}(t)} = -\gamma_{t_j}(t) \sum_{k=j+1}^{i-1} \frac{\alpha_k L(t, t_k, t_{k+1}) \gamma_{t_k}(t)}{1 + \alpha_k L(t, t_k, t_{k+1})} dt + \gamma_{t_j}(t) dW_1^{P_i}(t).$$

It is well-known that according to the Markov property the characteristic function:

$$\phi_0^i(u, y, v, l_0, \dots, l_{i-1}) = E^{P_i} \left(e^{iuY_i(t)} \mid Y_i(t_0) = y, V(t_0) = v, L_{t_0}(t_0) = l_0, \dots, L_{t_{i-1}}(t_0) = l_{i-1} \right)$$

is determined as the solution of a partial differential equation (PDE) that can be found through Feynman-Kac's theorem. To provide a closed-form solution in the spirit of Heston (1993), however, we need to ensure the linearity of the coefficients in the related PDE. This property is obviously destroyed by the presence of the drift correction term $\xi \sqrt{V(t)} \sigma(t, t_i) \rho_{14}$ in the dynamics of the volatility process due to the change of measure. It becomes immediately apparent that the only way to explicitly calculate the wanted characteristic function is by making the drift of the volatility process an affine

function of $V(t)$ and by eliminating the stochastic dependence on the LIBORs via the bond volatilities $\sigma(t, t_i)$ since the asymptotic form of the drift in the dynamics of the LIBORs rules out linearity with respect to $l_k, k = 0, \dots, i-1$. The classical approach to handling the LIBORs is by freezing them at their initial value. In addition, we need to approximate the square root of the volatility process within the drift function. Consequently, the dynamics of $V(t)$ become approximately of a square-root type and after redefining the system of SDEs (6):

$$\text{With } \sqrt{V(t)} \approx \frac{1}{2} \left(\sqrt{V(t_0)} + \frac{V(t)}{\sqrt{V(t_0)}} \right), dW_3^{P_i}(t) dW_4^{P_i}(t) = \rho_{34} dt \text{ and}$$

$$\sigma(t, t_i) \approx \sigma(t_0, t_i) = - \sum_{j=1}^{i-1} \frac{\alpha_j L_{t_j}(t_0)}{1 + \alpha_j L_{t_j}(t_0)} \gamma_{t_j}(t_0) - \frac{\alpha_0 L_{t_0}(t_0)}{1 + \alpha_0 L_{t_0}(t_0)} \gamma_{t_0}(t_0) \Rightarrow \tag{7}$$

$$dY_i(t) = -\frac{1}{2} \sigma_i^{fx}(t)^2 V(t) dt + \sigma_i^{fx}(t) \sqrt{V(t)} dW_3^{P_i}(t)$$

$$dV(t) = \left[\alpha(\theta - V(t)) + \xi \frac{1}{2} \left(\sqrt{V(t_0)} + \frac{V(t)}{\sqrt{V(t_0)}} \right) \sigma(t_0, t_i) \rho_{14} \right] dt + \xi \sqrt{V(t)} dW_4^{P_i}(t),$$

we come by the following PDE in the backward variables and the respective boundary condition:

$$\frac{\partial \phi}{\partial t_0} + \left[\alpha(\theta - v) + \frac{1}{2} \xi \sqrt{V(t_0)} \sigma(t_0, t_i) \rho_{14} + \frac{1}{2} \xi \frac{\sigma(t_0, t_i)}{\sqrt{V(t_0)}} \rho_{14} v \right] \frac{\partial \phi}{\partial v} - \frac{1}{2} \frac{\partial \phi}{\partial y} \sigma_i^{fx}(t_0)^2 v + \frac{1}{2} \frac{\partial^2 \phi}{\partial y^2} \sigma_i^{fx}(t_0)^2 v + \frac{1}{2} \frac{\partial^2 \phi}{\partial v^2} \xi^2 v + \frac{\partial^2 \phi}{\partial v \partial y} \sigma_i^{fx}(t_0) \xi \rho_{34} v = 0 \tag{8}$$

$$\phi_t^i(u) = e^{iuy}.$$

Suggested by the linearity of the PDE's coefficients in v , we propose a solution like:

$$\phi_t^i(u, y, v) = e^{C(t-t_0) + D(t-t_0)v + iuy}, \text{ where } C(0) = 0 \text{ and } D(0) = 0. \tag{9}$$

By plugging this ansatz into (8), we obtain two ordinary differential equations (ODEs):

$$\frac{\partial D}{\partial t_0} = \frac{1}{2} \xi^2 D^2 + \left[\frac{1}{2} \xi \frac{\sigma(t_0, t_i)}{\sqrt{V(t_0)}} \rho_{14} + iu \sigma_i^{fx}(t_0) \xi \rho_{34} - \alpha \right] D - \frac{1}{2} u^2 \sigma_i^{fx}(t_0)^2 - \frac{1}{2} iu \sigma_i^{fx}(t_0)^2 \tag{10}$$

$$\frac{\partial C}{\partial t_0} = D \left[\alpha \theta + \frac{1}{2} \xi \sqrt{V(t_0)} \sigma(t_0, t_i) \rho_{14} \right].$$

As shown in the appendix, the first one is a Riccati equation (see Oksendal (2000) Chapter 6) which can be solved by reducing it to a second-order linear ODE, whereas the second one is solved by direct integration. By means of their explicit solutions, as given by (B16) and (B18) respectively, we obtain an analytical expression for the characteristic function (9) that enables us to numerically determine the option price (5).

2.2. The calibration algorithm. For illustration purposes, we now consider a fictitious example of fitting the model to FX options across five different expiries and strikes. It is unnecessary to accentuate that the procedure can theoretically be extended to an arbitrarily wide range of maturities and strikes at the expense of rising computational time since the number of model parameters to match increases accordingly. Market prices of FX options in basis points, as displayed in Table 1, for strikes generated by

$$K_j(i) = FX_i(t_0) e^{0.1 \times \sqrt{i} \times \delta(j)}, \tag{11}$$

where $i, j = 1, \dots, 5$ and $\delta(j) = -1, -0.5, 0, 0.5, 1$

have been taken as a basis for the ensuing calibration routine.

Table 1. Market prices of FX calls (in bp) for different strikes and maturities

Expiry	FXopt 1	FXopt 2	FXopt 3	FXopt 4	FXopt 5
1y	74.31572	47.68933	25.64710	11.21897	3.99027
2y	90.20941	53.70243	23.56308	6.77430	1.26618
3y	98.96450	56.78179	21.40343	4.11798	0.38920
4y	103.73044	58.84490	20.92330	3.29027	0.20156
5y	105.93154	60.10471	21.14829	3.11624	0.15548

In the first place, we can check whether the FX option market is flat-smiled or not. The standard procedure would be to take the ATM strike, which happens to be that of the third FX option in the table above since $\delta(j) = 0$ and consequently $K_j(i) = FX_i(t_0)$, and to compute the ATM implied volatility by inversion of the lognormal option valuation formula. The same volatility is then utilized to price options with different strikes given by (11). The results, presented in Table 2, underline once again the fact that FX options exhibit a pronounced volatility skew with ITM options being underestimated whereas OTM ones are being systematically overestimated. From there the need stems to go beyond the standard geometric BROWNIAN motion and to resort to an extension of the lognormal-type dynamics of the forward forex rate that is based on stochastic volatility.

Table 2. FX call prices (in bp) computed with the ATM implied volatility

Expiry	FXopt 1	FXopt 2	FXopt 3	FXopt 4	FXopt 5
1y	71.97134	46.15283	25.64710	11.96459	4.55733
2y	88.39820	52.03995	23.56308	7.48320	1.54316
3y	97.65900	55.09986	21.40343	4.78883	0.53401
4y	102.85492	57.42792	20.92330	3.94375	0.31927
5y	105.22879	58.83967	21.14829	3.80896	0.28176

The calibration is performed on the FX call prices from Table 1, where the model prices are determined by (5). To obtain the best numerical estimate of both Fourier integrals, we employ the Gauss-Laguerre Quadrature, which is a Gaussian Quadra-

ture over the interval $[0, \infty)$ with a weighting function $w(x) = e^{-x}$ (see Abramowitz et al. (1972)). The model parameters to be calibrated are: (a) the parameters of the volatility process, initially set to $\xi = 0.01$, $\alpha = 0.02$, $\theta = 0.001$ and $V(t_0) = 0.01$; (b) the FX volatility coefficients, initially set to $\sigma_i^{fx}[i] = 1 - \exp(-0.05(i - 0.2))$, $i = 1, \dots, 5$; (c) the correlation coefficients, set to $\rho_{14} = \rho_{34} = -0.2$. Aiming to reproduce FX option values for a wide range of strikes and expiries, we solve the calibration problem by simultaneously varying the model parameters (a), (b) and (c) until the sum of squared basis point differences between model and market prices has been minimized, which, for the sake of completeness, is reported here to have been achieved at 8.56059 bp. We use a fast unconstrained non-linear minimization algorithm, the Davidon-Fletcher-Powell (DFP) conjugate gradient method as described in Press et al. (1996), to make the distance Δ^2 between the model and market matrices as small as possible:

$$\Delta^2 = \sum_{i,j=1}^5 \omega_{ij} (FXopt_{ij}^{market} - FXopt_{ij}^{model})^2 \rightarrow Min! \quad (12)$$

The calibration has been carried out with identical constant weights ω , which essentially corresponds to an attempt to obtain the best global fit to the “complete” market information, as represented by our market prices matrix. The resulting optimal solution is shown in Table 3 below.

Table 3. The best overall fit to the matrix of market call prices in Table 1

Expiry	FXopt 1	FXopt 2	FXopt 3	FXopt 4	FXopt 5
1y	73.44406	47.58264	26.25862	11.46718	3.54044
2y	89.40880	53.47140	24.14783	6.74194	0.80012
3y	98.25958	56.37568	21.92059	3.97862	0.14489
4y	103.30069	58.59928	21.35842	3.08668	0.05267
5y	105.40284	59.02163	19.80163	2.06586	0.00988

The stochastic volatility model brings about a significant improvement over the deterministic volatility one in any case. Prices of mid-maturity FX calls (i.e., 3y and 4y) are reproduced with a very good precision. For very short- and long-maturity options (i.e., 5y), however, the goodness of fitting worsens suggesting that the assumption of a unique volatility process $V(t)$ common to all forward forex rates might be too restrictive, especially when calibrating to a very wide range of maturities.

In order to capture certain features of a given exotic instrument, one could alternatively try to achieve the best fit to only a specific portion of the matrix of market prices sacrificing the remaining part of it. The quality of the partial calibration will be governed by the, in this case, non-constant weights ω assigned to the elements of the Δ^2 distance function. The choice of the weights will mostly depend on the particular pricing problem.

Conclusions

We proposed an integrated Cross-Currency LIBOR Market Model under a uniform probability measure in a multifactorial environment. The chief purpose of our paper has been the development of a viable pricing framework for exotic cross-currency interest rate instruments that is at the same time flexible enough to allow for the incorporation of as much available market information as possible. In terms of FX markets, on which the main focus of this work has been placed, fulfilling this purpose in a satisfying manner required the calibration to the whole range of FX options prices across all available maturities and strikes. This line of modelling has been reinforced by the significant volatility skew typically observed with FX options and the fact that it seems impossible for the most cross-currency derivatives to choose a particular strike, or a specific expiry of an FX option since they usually depend on a variety of strikes and maturities. The procedure eventually culminated in an extension of the lognormal-type dynamics of the forward forex rate beyond the geometric BROWNIAN motion and the postulation of a stochastic volatility evolution in conformity with Heston (1993) based on a unique volatility, common to all forward FX rates, that follows a mean-reverting square-root process. After determining an analytical expression for the conditional characteristic function in a mildly approximate fashion for the sake of mathematical tractability, closed-form formulas for FX option prices have been obtained and numerically estimated with the aid of the Gauss-Laguerre Quadrature. The model prices computed in this manner served as the backbone of the ensuing calibration routine, by means of which we matched the model parameters so that the distance between the model and market prices matrices has been minimized. We have seen that introducing stochastic volatility led to a substantial improvement over the deterministic model in any case, although the fitting quality slightly worsened for very short- and especially long-dated options as compared to mid-maturity ones. This feature has been ascribed to the possibility of our unique volatility process common to all forward FX rates being too restrictive, especially when faced with a wide maturity spectrum to

calibrate to. More flexibility could be introduced by considering a different stochastic volatility process for the dynamics of each forward FX rate, however, inevitably making the calibration more cumbersome because of the additional volatility and correlation parameters and raising the potential problem of overfitting the model due to the increased number of

parameters. Above all, the pricing of cross-currency exotic interest rate products would become a very difficult task since the drift functions within the dynamics of both the foreign LIBOR and the forward forex rate would, aside from the unpleasant stochastic dependence on LIBOR rates, involve extra intra- and intercorrelated volatility processes.

References

1. Abramowitz, M. and I. A. Stegun (1972). Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, Dover, New York, pp. 890 and 923.
2. Andersen, L., and J. Andreasen (2000). "Volatility Skews and Extensions of the LIBOR Market Model", Applied Mathematical Finance, 7, 1-32.
3. Andersen, L., and J. Andreasen (2002). "Volatile Volatilities", RISK December, 163-168.
4. Andersen, L. and R. Brotherton-Ratcliffe (2005). "Extended LIBOR Market Models with Stochastic Volatility", The Journal of Computational Finance, 9, 1.
5. Bakshi, G., and D. Madan (2000). "Spanning and Derivative Security Valuation", Journal of Financial Economics, 55, 205-238.
6. Benner, W., Zypkov, L., and S. Jortzik (2007). "A Multifactorial Cross-Currency LIBOR Market Model", SSRN working paper.
7. Brigo, D., and F. Mercurio (2002). "Calibrating Libor," RISK January, 117-121.
8. Carr, P., and D. Madan (1999). "Option Valuation using the Fast Fourier Transform", Journal of Computational Finance, 2, 61-73.
9. Daniluk, A., and D. Gatarek (2005). "A fully lognormal LIBOR Market Model", RISK September, 115-119.
10. Heston, S. (1993). "A Closed-Form Solution for Options with Stochastic Volatility with Applications to Bond and Currency Options", The Review of Financial Studies, 6, 2, 327-343.
11. Gil-Pelaez, J. (1951). "Note on the Inversion Theorem", Biometrika, 38, 3/4, 481-482.
12. Glasserman, P., and N. Merener, 2003a, "Numerical Solution of Jump-Diffusion LIBOR Market Models", Finance and Stochastics, 7, 1, 1-27.
13. Glasserman, P. and S.G. Kou (2003b). "The Term Structure of Simple Forward Rates with Jump Risk", Mathematical Finance, 13, 3, 383-410.
14. Joshi, M., and R. Rebonato (2003). "A Displaced-Diffusion Stochastic Volatility LIBOR Market Model: Motivation, Definition and Implementation", Quantitative Finance, 3, 6, 458-469.
15. Oksendal, B. (2000). Stochastic Differential Equations. An Introduction with Applications, Springer Verlag, Berlin Heidelberg New York.
16. Piterbarg, V. (2003). "A Stochastic Volatility Forward LIBOR Model with a Term Structure of Volatility Smiles", SSRN Working Paper.
17. Piterbarg, V. (2005a). "Stochastic Volatility Model with Time-Dependent Skew", Applied Mathematical Finance, 12, 2, 147-185.
18. Piterbarg, V. (2005b). "Time to Smile", RISK May, 71-75.
19. Piterbarg, V. (2006). "Smiling Hybrids", RISK May, 66-71.
20. Press, W.H., Teukolsky, S.A., Vetterling, W.T., and B.P. Flannery (1996). Numerical Recipes in C. The Art of Scientific Computing, Cambridge University Press, Cambridge, pp. 425-430.
21. Rebonato, R. (2002). Modern Pricing of Interest Rate Derivatives. The LIBOR Market Model and Beyond, Princeton University Press, New Jersey.
22. Rubinstein, M. (1983). "Displaced Diffusion Option Pricing", Journal of Finance, 38, 213-217.
23. Schloegl, E. (2002). "A Multicurrency Extension of the Lognormal Interest Rate Market Models", Finance and Stochastics, 6, 173-196.
24. Schoenmakers, J., and B. Coffey, 2003, "Systematic Generation of Correlation Structures for the LIBOR Market Model", International Journal of Theoretical and Applied Finance, 6, 4, 1-13.
25. Scott, L. (1997). "Pricing Stock Options in a Jump-Diffusion Model with Stochastic Volatility and Interest Rates: Application of Fourier Inversion Methods", Mathematical Finance, 7, 413-426.

Appendix A. FX option pricing formula

Based on a derivation of the inversion theorem by Gil-Pelaez (1951), we determine the probability of finishing in-the-money, where $\ln(FX_i(t)) = Y_i(t)$ and $\ln(K) = k$, as follows:

$$\text{With } \phi_{t_0}^i(u) = \int_{-\infty}^{\infty} e^{iuY} dP(Y) = \int_{-\infty}^{\infty} (\cos(uY) + i \sin(uY)) dP(Y),$$

$$\frac{e^{iuk} \phi_{t_0}^i(-u) - e^{-iuk} \phi_{t_0}^i(u)}{iu} = \frac{2i \sin(uk) \int_{-\infty}^{\infty} \cos(uY) dP(Y) - 2 \cos(uk) \int_{-\infty}^{\infty} i \sin(uY) dP(Y)}{iu}$$

and $\operatorname{Re} \left[\frac{e^{-iuk} \phi_{t_0}^i(u)}{iu} \right] = \frac{\cos(uk) \int_{-\infty}^{\infty} i \sin(uY) dP(Y) - i \sin(uk) \int_{-\infty}^{\infty} \cos(uY) dP(Y)}{iu} \Rightarrow$

$$P\{Y_i(t) < k\} = \frac{1}{2} + \frac{1}{2\pi} \int_0^{\infty} \frac{e^{iuk} \phi_{t_0}^i(-u) - e^{-iuk} \phi_{t_0}^i(u)}{iu} du = \frac{1}{2} - \frac{1}{\pi} \int_0^{\infty} \operatorname{Re} \left[\frac{e^{-iuk} \phi_{t_0}^i(u)}{iu} \right] du \Rightarrow$$

$$P\{Y_i(t) > k\} = 1 - P\{Y_i(t) < k\} = \frac{1}{2} + \frac{1}{\pi} \int_0^{\infty} \operatorname{Re} \left[\frac{e^{-iuk} \phi_{t_0}^i(u)}{iu} \right] du \tag{A12}$$

The delta of the option is somewhat convoluted, though it can be determined by similar techniques. For any positive numbers λ and ε , we have:

$$\frac{1}{\pi} \int_{\varepsilon}^{\lambda} \operatorname{Re} \left[\frac{e^{-iuk} \phi_{t_0}^i(u-i)}{iu} \right] du = \frac{1}{\pi} \int_{\varepsilon}^{\lambda} \int_{-\infty}^{\infty} \operatorname{Re} \left[\frac{e^{-iuk} e^{i(u-i)Y} dP(Y)}{iu} \right] du = \frac{1}{\pi} \int_{\varepsilon}^{\lambda} \int_{-\infty}^{\infty} \operatorname{Re} \left[\frac{e^Y e^{iu(Y-k)} dP(Y)}{iu} \right] du =$$

$$\frac{1}{\pi} \int_{-\infty}^{\infty} e^Y dP(Y) \int_{\varepsilon}^{\lambda} \operatorname{Re} \left[\frac{\cos u(Y-k) + i \sin u(Y-k)}{iu} \right] du = \frac{1}{\pi} \int_{-\infty}^{\infty} e^Y dP(Y) \int_{\varepsilon}^{\lambda} \frac{\sin u(Y-k)}{u} du$$

Proceeding from this expression now, we obtain the step function by simply letting ε tend to zero and λ tend to infinity as shown in Gil-Pelaez (1951). Thereafter, we conveniently arrive at the wanted conditional expectation in the following manner:

$$\frac{1}{\pi} \int_{-\infty}^{\infty} e^Y dP(Y) \lim_{\substack{\lambda \rightarrow \infty \\ \varepsilon \rightarrow 0}} \int_{\varepsilon}^{\lambda} \frac{\sin u(Y-k)}{u} du = \frac{1}{2} \int_{-\infty}^{\infty} \operatorname{sign}(Y-k) e^Y dP(Y) = \frac{1}{2} \int_{Y>k} e^Y dP(Y) - \frac{1}{2} \int_{Y<k} e^Y dP(Y)$$

With $\int_{Y>k} e^Y dP(Y) = \phi_{t_0}^i(-i) \Rightarrow \int_{Y>k} e^Y dP(Y) = \frac{\phi_{t_0}^i(-i)}{2} + \frac{1}{\pi} \int_0^{\infty} \operatorname{Re} \left[\frac{e^{-iuk} \phi_{t_0}^i(u-i)}{iu} \right] du,$

essentially meaning that the delta of the option is defined by:

$$\frac{1}{2} + \frac{1}{\pi} \int_0^{\infty} \operatorname{Re} \left[\frac{e^{-iuk} \phi_{t_0}^i(u-i)}{iu \phi_{t_0}^i(-i)} \right] du \tag{A13}$$

Thus, additionally observing that $\phi_{t_0}^i(-i) = E^{P_{t_0}} \left(e^{i(-i)Y_i(t)} \mid F_{t_0} \right) = FX_i(t_0)$, the option price is readily computed in terms of the conditional characteristic function like:

$$\frac{FX_{opt}_i(t_0)}{B(t_0, t_i)} = E^{P_{t_0}} \left[(FX_i(t) - K)^+ \mid F_{t_0} \right]$$

$$= \left(\frac{\phi_{t_0}^i(-i)}{2} + \frac{1}{\pi} \int_0^{\infty} \operatorname{Re} \left[\frac{e^{-iuk} \phi_{t_0}^i(u-i)}{iu} \right] du \right) - K \left(\frac{1}{2} + \frac{1}{\pi} \int_0^{\infty} \operatorname{Re} \left[\frac{e^{-iuk} \phi_{t_0}^i(u)}{iu} \right] du \right) \tag{A14}$$

$$= FX_i(t_0) \left(\frac{1}{2} + \frac{1}{\pi} \int_0^{\infty} \operatorname{Re} \left[\frac{e^{-iuk} \phi_{t_0}^i(u-i)}{iu \phi_{t_0}^i(-i)} \right] du \right) - K \left(\frac{1}{2} + \frac{1}{\pi} \int_0^{\infty} \operatorname{Re} \left[\frac{e^{-iuk} \phi_{t_0}^i(u)}{iu} \right] du \right).$$

Appendix B. Solution of the ordinary differential equations

Starting with the Riccati equation in (10):

$$\frac{\partial D}{\partial t_0} = aD^2 + bD + c, \quad \text{with } a = \frac{1}{2}\xi^2, \quad b = \frac{1}{2}\xi \frac{\sigma(t_0, t_i)}{\sqrt{V(t_0)}} \rho_{14} + iu\sigma_i^{fx}(t_0)\xi\rho_{34} - \alpha, \tag{B15}$$

$$c = -\frac{1}{2}u^2\sigma_i^{fx}(t_0)^2 - \frac{1}{2}iu\sigma_i^{fx}(t_0)^2,$$

it has been led back to a second-order linear differential equation by substitution:

$$\text{Put } aD = v \Rightarrow v' = aD' \Rightarrow v' = v^2 + bv + ca$$

$$\text{Substitute } v = \frac{u'}{u} \text{ and since } v' = -\frac{u''}{u} + v^2 \Rightarrow u'' = -bu' - acu.$$

The solution ansatz is of an exponential type and is plugged into the ODE to be solved:

$$e^{z(t-t_0)} \Rightarrow z^2 - bz + ac = 0 \Rightarrow z_{1,2} = \frac{b \pm \sqrt{b^2 - 4ac}}{2},$$

ultimately arriving at the following general solution along with the respective boundary condition:

$$Ae^{\frac{b+\sqrt{b^2-4ac}}{2}(t-t_0)} + Be^{\frac{b-\sqrt{b^2-4ac}}{2}(t-t_0)}$$

$$\text{From } D(0) = 0 \text{ and } D = \frac{u'}{au} \Rightarrow u'(t-t_0)|_{t=t_0} = 0 \Rightarrow$$

$$A \frac{b + \sqrt{b^2 - 4ac}}{2} + B \frac{b - \sqrt{b^2 - 4ac}}{2} = 0.$$

However, we are not aimed at finding an explicit solution of the second-order linear differential equation. We do not need to compute A and B separately, which is by the way based on a single boundary condition impossible, the ratio A/B would suffice since we actually seek to determine $D(t-t_0)$. Having made this crucial observation, we obtain:

$$\text{With } \frac{A}{B} = -\frac{b - \sqrt{b^2 - 4ac}}{b + \sqrt{b^2 - 4ac}} \Rightarrow$$

$$\begin{aligned} D(t-t_0) &= \frac{u'}{au} = \frac{A \frac{b + \sqrt{b^2 - 4ac}}{2} e^{\frac{b + \sqrt{b^2 - 4ac}}{2}(t-t_0)} + B \frac{b - \sqrt{b^2 - 4ac}}{2} e^{\frac{b - \sqrt{b^2 - 4ac}}{2}(t-t_0)}}{aAe^{\frac{b + \sqrt{b^2 - 4ac}}{2}(t-t_0)} + aBe^{\frac{b - \sqrt{b^2 - 4ac}}{2}(t-t_0)}} \Big/ : B \\ &= \frac{\frac{-b + \sqrt{b^2 - 4ac}}{2} e^{\frac{b + \sqrt{b^2 - 4ac}}{2}(t-t_0)} + \frac{b - \sqrt{b^2 - 4ac}}{2} e^{\frac{b - \sqrt{b^2 - 4ac}}{2}(t-t_0)}}{a \frac{-b + \sqrt{b^2 - 4ac}}{b + \sqrt{b^2 - 4ac}} e^{\frac{b + \sqrt{b^2 - 4ac}}{2}(t-t_0)} + ae^{\frac{b - \sqrt{b^2 - 4ac}}{2}(t-t_0)}} \Big/ : e^{\frac{b - \sqrt{b^2 - 4ac}}{2}(t-t_0)}} \\ &= \frac{\frac{-b + \sqrt{b^2 - 4ac}}{2} + \frac{b - \sqrt{b^2 - 4ac}}{2} e^{\sqrt{b^2 - 4ac}(t-t_0)}}{a + a \frac{-b + \sqrt{b^2 - 4ac}}{b + \sqrt{b^2 - 4ac}} e^{\sqrt{b^2 - 4ac}(t-t_0)}} \Rightarrow \end{aligned}$$

$$D(t-t_0) = \frac{\sqrt{b^2 - 4ac} - b}{2a} \left(\frac{1 - e^{\sqrt{b^2 - 4ac}(t-t_0)}}{1 - \frac{b - \sqrt{b^2 - 4ac}}{b + \sqrt{b^2 - 4ac}} e^{\sqrt{b^2 - 4ac}(t-t_0)}} \right). \tag{B16}$$

The second ODE:

$$\frac{\partial C}{\partial t_0} = \alpha_{-}\theta.D, \text{ with } \alpha_{-}\theta = \alpha\theta + \frac{1}{2}\xi\sqrt{V(t_0)}\sigma(t_0, t_i)\rho_{14} \tag{B17}$$

is solved by direct integration:

$$\text{With } C(0) = 0 \text{ and } j = \sqrt{b^2 - 4ac} \Rightarrow C(0) - C(t - t_0) = \alpha_{-}\theta \int_{t_0}^t D(t - u)du \Rightarrow$$

$$C(t - t_0) = \alpha_{-}\theta \int_0^{t-t_0} D(u)du = \alpha_{-}\theta \frac{j-b}{2a} \int_0^{t-t_0} \frac{1 - e^{ju}}{1 - \frac{b-j}{b+j}e^{ju}} du. \text{ Put } 1 - \frac{b-j}{b+j}e^{ju} = y \Rightarrow$$

$$\begin{aligned} C(t - t_0) &= -\alpha_{-}\theta \frac{j-b}{2aj} \int_{1 - \frac{b-j}{b+j}}^{1 - \frac{b-j}{b+j}e^{j(t-t_0)}} \frac{1 - \frac{(1-y)(b+j)}{b-j}}{y(1-y)} dy \\ &= \alpha_{-}\theta \frac{j-b}{2aj} \int_{1 - \frac{b-j}{b+j}}^{1 - \frac{b-j}{b+j}e^{j(t-t_0)}} \frac{b+j}{y(b-j)} dy - \alpha_{-}\theta \frac{j-b}{2aj} \int_{1 - \frac{b-j}{b+j}}^{1 - \frac{b-j}{b+j}e^{j(t-t_0)}} \frac{1}{y(1-y)} dy \\ &= \alpha_{-}\theta \frac{j-b}{2aj} \frac{b+j}{b-j} \ln \left(\frac{1 - \frac{b-j}{b+j}e^{j(t-t_0)}}{1 - \frac{b-j}{b+j}} \right) - \alpha_{-}\theta \frac{j-b}{2aj} \int_{1 - \frac{b-j}{b+j}}^{1 - \frac{b-j}{b+j}e^{j(t-t_0)}} \left(\frac{1}{y} + \frac{1}{1-y} \right) dy \\ &= -\frac{\alpha_{-}\theta(b+j)}{2aj} \ln \left(\frac{1 - \frac{b-j}{b+j}e^{j(t-t_0)}}{1 - \frac{b-j}{b+j}} \right) - \frac{\alpha_{-}\theta(j-b)}{2aj} \ln \left(\frac{1 - \frac{b-j}{b+j}e^{j(t-t_0)}}{1 - \frac{b-j}{b+j}} \right) \\ &\quad + \alpha_{-}\theta \frac{j-b}{2aj} \int_{1 - \frac{b-j}{b+j}}^{1 - \frac{b-j}{b+j}e^{j(t-t_0)}} \frac{1}{1-y} d(1-y) \\ &= -\frac{\alpha_{-}\theta}{a} \ln \left(\frac{1 - \frac{b-j}{b+j}e^{j(t-t_0)}}{1 - \frac{b-j}{b+j}} \right) + \alpha_{-}\theta \frac{j-b}{2aj} \int_{1 - \frac{b-j}{b+j}}^{1 - \frac{b-j}{b+j}e^{j(t-t_0)}} \frac{1}{1-y} d(1-y) \Rightarrow \\ C(t - t_0) &= -\frac{\alpha_{-}\theta}{a} \ln \left(\frac{1 - \frac{b-j}{b+j}e^{j(t-t_0)}}{1 - \frac{b-j}{b+j}} \right) + \alpha_{-}\theta \frac{j-b}{2aj} (t - t_0). \tag{B18} \end{aligned}$$

In conclusion, we hold (B16) and (B18) to be the solutions of the ordinary differential equations in (10) being integral part of the characteristic function (9), whereby a , b , c and $\alpha_{-}\theta$ are the substitutes defined previously by (B15) and (B17) respectively.