Measuring Arbitrage Profits in Imperfect Markets[1]

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Abstract

In this paper we introduce some optimization problems that provide us with a measure testing the degree of efficiency in securities markets with bid-ask spreads. The measure tests relative arbitrage profits when there are transaction costs on the prices and payoffs of the assets. Moreover, we prove that the measure is the minimum of the measures of efficiency in all frictionless markets where the prices and payoffs lie between the bid and the ask prices and payoffs respectively. In particular, we find that the model is arbitrage-free if and only if there exist convex combinations of the bid and the ask prices and payoffs such that the corresponding frictionless model is arbitrage-free.

Key words: Arbitrage Measurement, Imperfect Market, State Price.

Introduction

This paper deals with two topics usual in finance: the arbitrage measurement, and the relationship between markets with transaction costs and some underlying frictionless markets.

The arbitrage measurement is the key to establish the level of integration of two or more financial markets. So, two perfectly integrated markets give the same price to identical payoffs and, moreover, no cross-market arbitrage strategies can be implemented. On the contrary, the lack of integration causes the existence of arbitrage opportunities.

Many empirical papers (Kleidon and Whaley (1992), Kamara and Miller (1995), Kempf and Korn (1998), etc.) analyze the existence of cross-market arbitrage and frequently conclude that the arbitrage seems to occur, although imperfections make it difficult to decide if the arbitrage profits may be obtained after discounting the transaction costs. Chen and Knez (1995) develope a general measure of cross-market integration, and when they empirically test the measure, it seems to demonstrate the existence of arbitrage. However, as pointed out by the authors, this result seems to be very sensitive with respect to the frictionless assumptions.

Balbás and Muñoz (1998) introduce a new measure that computes relative arbitrage gains and is able to discount some special kinds of transaction costs. As pointed out by Balbás *et al.* (2000), the measure may be easily computed in many empirical studies and provides useful procedures to test the level of integration of several financial markets. Moreover, the measure may also apply to price and hedge new derivative securities (see Balbás *et al.* (1999b) for further details on this point).

We will follow here the ideas of Balbás and Muñoz (1998) and extend the analysis in order to involve general transaction costs. The extension allows empirical and practical applications of the measure to increase, and yields new theoretical results concerning imperfect financial markets.

In order to incorporate general transaction costs, we will consider the approach of Jouini and Kallal (1995) and, consequently, it will be assumed the existence of two prices and payoffs per security. Obviously, prices (payoffs) will be larger (lower) when traders buy, and lower (larger) if they sell.

Paper's outline is as follows: the second section introduces the basic notations and concepts. The measure of arbitrage is defined in the third section. Following Balbás and Muñoz (1998), the measure provides relative arbitrage profits with respect to the price of the sold assets. It is nonnegative and it vanishes if and only if the model is arbitrage free. As it will be proved, this measure also enables us to determine the maximum relative arbitrage profit with respect to the total traded value (price of the sold and purchased assets). This interpretation allows us to discount some types of transaction costs that might be difficult to include in the usual bid-ask spread models. Although the

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measure maximizes relative arbitrage profits and, consequently, a non linear ratio, it may be easily computed in practice since an equivalent linear optimization problem is provided. The linear problem also leads to the optimal arbitrage portfolio. The section ends by showing the continuity of the measure with respect to the initial parameters and data. It is a very important property because the measure is not sensitive with respect to several types of assumptions or errors committed when computing the data. In particular, the measure may also apply to test the degree of fulfilment in practice of theoretical asset pricing models, since the procedure proposed by Hansen and Jagannathan (1997) (these authors use the Chen and Knez (1995) measure) may be adapted.

The fourth section presents a dual linear optimization problem that also leads to the measure and is useful for several reasons. In fact, it provides new interpretations for the measure, a proxy of "state prices" (in the sense of Chamberlain and Rothschild (1983)) even for no arbitrage free economies and two characterizations of the arbitrage absence in the imperfect market case. So, the measure vanishes (the model is arbitrage free) if and only if there are state prices (probability measures) such that the best (maximum) expected return associated to long positions is worse (lower) than the best (minimum) expected return associated to short positions. Moreover, it is also equivalent to the existence of an arbitrage free frictionless model for which prices and payoffs lie within the spreads. This result is in line with the one of Jouini and Kallal (1995). They extend the martingale property of Harrison and Kreps (1979) to a market with bid-ask price processes. They found that the no existence of free-lunch in their model corresponds to the no existence of a free-lunch in a frictionless price process lying between the bid and the ask processes.

This is also related to some results of Pham and Touzi (1999) since they proved that for a model where transaction costs are linear the no existence of free lunch is equivalent to the absence of arbitrage. Then, applying the results of Jouini and Kallal (1995), they obtain a similar property to the absence of arbitrage in a market with linear transaction costs.

When the measure is strictly positive, the dual problem yields a proxy for the state prices that leads to a new interpretation for the measure. It represents minimum relative (per dollar) errors committed by agents when they give bid and ask prices for the available securities.

The developed theory may also be adapted so that it can apply to bond markets, in the line of Jaschke (1998). In such a case, our results will yield new characterizations of the arbitrage absence in a bond market such that coupons associated to long positions are lower than coupons associated to short positions (due to taxes, for instance). Moreover, a term structure of interest rates (or its proxy) may be introduced for this bond market, even if it is not arbitrage free.

It should be first pointed out that we have modeled the underlying uncertainty (the states of the world) by compact spaces (rather than L^2-spaces). For an empirical implementation this may be more convenient since we do not require any initial probability measure in the set of states of the world. Second, we have only considered arbitrage portfolios of the second type (in the sense of Ingersoll (1987)). However, most of our results (but not all of them) hold in a L^2-space and also concerning the arbitrage in the usual sense.

Preliminaries

Consider an economy endowed with a Hausdorff compact topological space K, on which the linear space $C(K)$ of all continuous functions over the real line $\mathbb{R}$ is defined. When equipped with the norm

$$\|\alpha\| = Sup\ \{ |\alpha(k)| : k\in K\}$$

for any $\alpha\in C(K)$, the space $M(K)$ of Radon measures over K is known to be the dual space of $C(K)$ (Riesz representation Theorem). Here we are assuming that K is the set of outcome states and for some $\alpha\in C(K)$, $\alpha(k)$ represents the payoff of a portfolio in the state of nature k for every $k\in K$. This restriction to continuous contingent claims is made for expositional and mathematical ease.

Let the number of assets be finite and indexed by $\{1,2,\ldots,n\}$. Each security i with $i=1,2,\ldots,n$, can be bought for its ask price a_i, and can be sold for its bid price b_i at the initial date. The payoff on the i-th asset in the second period is given by $B_i\in C(K)$ for a bought security and by

$A_i \in C(K)$ for a sold security. We assume that $a_i \geq b_i > 0$, $A_i(k) \geq B_i(k)$, and $A_1(k) \geq B_1(k) > 0$, for every $k \in K$. For a portfolio $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, the sum

$$P(x)(k) = x^+_1 B_1 + x^+_2 B_2 \ldots + x^+_n B_n - x^-_1 A_1 - x^-_2 A_2 - \ldots - x^-_n A_n$$

is its total payoff and

$$p(x) = x^+_1 a_1 + x^+_2 a_2 \ldots + x^+_n a_n - x^-_1 b_1 - x^-_2 b_2 - \ldots - x^-_n b_n$$

is its current price[1].

Definition 1. The portfolio $x \in \mathbb{R}^n$ is said to be an arbitrage opportunity (strong form) if $P(x)(k) \geq 0$ for every $k \in K$ and $p(x) < 0$.

Thus, an arbitrage opportunity allows an agent to increase consumption in the initial period and at least not to decrease consumption in the second period. We do not consider here arbitrage opportunities of first type. Consequently, it is not true that absence of arbitrage opportunities in our model implies the absence of arbitrage of both types or the absence of free lunch. The measure defined below captures only the existence of arbitrage opportunities as in Definition 1[2].

Arbitrage Measurement

In order to measure arbitrage profits we can do it in relative terms. We look for a portfolio minimizing the initial investment needed to get a nonnegative payoff in every state of nature and with total sold assets price at most one unity, *i.e.*,

$$\begin{gathered} Maximize\ -p(x) \\ P(x)(k) \geq 0, \text{ for every } k \in K \\ x^-_1 b_1 + x^-_2 b_2 + \ldots + x^-_n b_n \leq 1. \end{gathered}$$

The latter non linear optimization problem can be easily transformed in a linear one by considering a strategy as a pair $(x,y) \in \mathbb{R}^n$ of long and short non negative components, *i.e.*, x_i denotes the quantity of the *i-th* security bought and y_i denotes the quantity of the *i-th* security sold. Then, we obtain the following problem:

$$\begin{gathered} Maximize\ -x_1 a_1 - x_2 a_2 \ldots - x_n a_n + y_1 b_1 + y_2 b_2 + \ldots + y_n b_n \\ x_1 B_1(k) + x_2 B_2(k) + \ldots + x_n B_n(k) - y_1 A_1(k) - y_2 A_2(k) - \ldots - y_n A_n(k) \geq 0,\ k \in K \qquad (1) \\ y_1 b_1 + y_2 b_2 + \ldots + y_n b_n \leq 1 \\ x_i \geq 0,\ y_i \geq 0. \end{gathered}$$

Some remarks are in order to ensure that latter problem is solvable, *i.e.*, that the maximum arbitrage profit is available.

First, we do not impose the constraints $x_i y_i = 0$ since for every feasible (x,y) the pair $((x-y)^+, (x-y)^-)$ is also feasible and with a current price at most the one of (x,y).

Second, the problem is consistent since $(x,y) = (0,0)$ is feasible. Consequently, the optimum value of Problem (1) is nonnegative. Moreover, the value of Problem (1) is bounded by $y_1 b_1 + y_2 b_2 + \ldots + y_n b_n \leq 1$ and hence finite.

The whole feasible set of Problem (1) is not a bounded one, but adding the constraint $x_1 a_1 + x_2 a_2 \ldots + x_n a_n - y_1 b_1 - y_2 b_2 - \ldots - y_n b_n \leq 0$ we obtain a bounded subset of the feasible set containing the optimal solution. We then get a problem whose feasible set is a compact one and with the same optimal solution. Since the objective function is continuous, we derive that the optimal value $\boldsymbol{m}$ of Problem (1) is attained by a feasible (x,y) and it verifies the inequalities $0 \leq \boldsymbol{m} \leq 1$.

Definition 2. We define the measure $\boldsymbol{m}$ of arbitrage opportunities as the optimum value achieved in Problem (1).

One can easily check that the definition of $\boldsymbol{m}$ is consistent in the following sense:

Theorem 1. No arbitrage opportunity exists on the market if and only if $\boldsymbol{m} = 0$.

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[1] As usual, $x^+ = Max\ \{x,0\}$ and $x^- = Max\ \{-x,0\}$ for every $x \in \mathbb{R}$.

[2] It is possible to incorporate some modifications in order to measure arbitrage of the first type. However, the procedure is not straightforward and is beyond the scope of this paper.

In the particular case when there are no transaction costs, $B_i = A_i$ and $b_i = a_i$, the measure $\boldsymbol{m}$ is just an extension of the measure of the degree of the fulfilment of the Law of One Price of Balbás and Muñoz (1998). Here we test some arbitrage profits (in the strong form) even when the Law of One Price holds. Denoting by $\boldsymbol{m}_{p,\alpha}$ the frictionless measure of arbitrage opportunities where $p \in \mathbb{R}^n$ and $\alpha \in C(K)^n$ are such that $b_i \le p_i \le a_i$ and $B_i \le \alpha_i \le A_i$, we show in the following section how the measures $\boldsymbol{m}$ and $\boldsymbol{m}_{p,\alpha}$ are related.

It is possible to show (as in Balbás and Muñoz (1998) for the frictionless case) that the measure $\boldsymbol{m}$ has different and interesting interpretations. If we define the relative profit functions f and g in $\mathbb{R}^{2n}$ by

$$f(x,y)= (-x_1a_1-x_2a_2 \ldots -x_na_n +y_1b_1 + y_2b_2 +\ldots+ y_nb_n)/(y_1b_1 + y_2b_2 +\ldots+ y_nb_n)$$

and

$$g(x,y) = (-x_1a_1-x_2a_2 \ldots -x_na_n +y_1b_1 + y_2b_2 +\ldots+ y_nb_n)/(x_1a_1+x_2a_2 \ldots +x_na_n +y_1b_1 + y_2b_2 +\ldots+ y_nb_n)$$

with the convention that $f(0,0) = g(0,0) = 0$, it is easily proved that $g(x,y) = f(x,y)/(2 - f(x,y))$ whenever $y \neq 0$ and the following theorem:

Theorem 2. Let (x^*,y^*) be feasible in (1) and such that

$$\boldsymbol{m} = -x^*_1a_1-x^*_2a_2 \ldots -x^*_na_n +y^*_1b_1 + y^*_2b_2 +\ldots+ y^*_nb_n$$

1. (x^*,y^*) solves the problem

$$\text{Maximize } f(x,y)$$
$$x_1B_1(k)+x_2B_2 (k)+\ldots+x_nB_n (k) - y_1A_1 (k) - y_2A_2 (k)-\ldots-y_nA_n(k) \ge 0,\ k \in K$$

and the equality $\boldsymbol{m} = f(x^*,y^*)$ holds.

2. (x^*,y^*) solves the problem

$$\text{Maximize } g(x,y)$$
$$x_1B_1(k)+x_2B_2 (k)+\ldots+x_nB_n (k) - y_1A_1 (k) - y_2A_2 (k)-\ldots-y_nA_n(k) \ge 0,\ k \in K$$

and the equality $\boldsymbol{m}/(2-\boldsymbol{m}) = g(x^*,y^*)$ holds.

The first statement establishes that $\boldsymbol{m}$ can be seen as the maximum arbitrage profit in relation to the price of all the sold assets while *ii)* states that the same portfolio leading to the measure $\boldsymbol{m}$ also maximizes the arbitrage profit $\boldsymbol{l}$ in relation to the price of all interchanged assets. This is significant since, once computing this maximum profit, it allows us to incorporate other transaction costs than bid-ask spreads. In particular, those frictions that are determined by the price of the exchanged assets.

The theorem above is also useful to prove in an easy way that the arbitrage measure $\boldsymbol{m}$ is a continuous function with respect to the bid and ask prices of traded securities. This is important in computing $\boldsymbol{m}$ in empirical applications. More precisely, let Π and Γ be the following sets:

$$\Pi = \{(b,a) \in \mathbb{R}^{2n}\ /0 < b_i \le a_i,\ i=1,2,\ldots,n\}$$

$$\Gamma =\{(B,A) \in C(K)^{2n}\ /B_1(k) > 0,\ B_i(k) \le A_i(k),\ k\in K,\ i=1,2,\ldots,n \}.$$

Considering the function $m(b,a,B,A)$ defined from $\Pi x\Gamma$ to $\mathbb{R}$ and equipping $\Pi x\Gamma$ with the product topology ($C(K)$ is endowed with the supremum norm) we get:

Theorem 3. The measure $\boldsymbol{m}$ is continuous on $\Pi x\Gamma$.

Proof. Let $(b^j,a^j,B^j,A^j)_{j\in N}$ be a sequence in $\Pi x\Gamma$ converging to $(b^*,a^*,B^*,A^*) \in \Pi x\Gamma$. Set $\boldsymbol{m}^j = m(b^j,a^j,B^j,A^j)$ and $\boldsymbol{m} = m(b^*,a^*,B^*,A^*)$. We denote by Problem ($1^j$) and Problem (1) the corresponding problems with prices (b^j,a^j,B^j,A^j) and (b^*,a^*,B^*,A^*) respectively. Take (x^j,y^j) feasible in

(1^j) and such that

$$m^j = -x^j_1 a^j_1 - x^j_2 a^j_2 \ldots - x^j_n a^j_n + y^j_1 b^j_1 + y^j_2 b^j_2 + \ldots + y^j_n b^j_n$$

for every natural j. From $m^j \geq 0$ and $y^j_1 b^j_1 + y^j_2 b^j_2 + \ldots + y^j_n b^j_n \leq 1$ it can be deduced that $0 \leq x^j_i \leq (1/a^j_i)$ and $0 \leq y^j_i \leq (1/b^j_i)$. Thus $(x^j, y^j)_{j \in N}$ is a bounded sequence in $\mathbb{R}^{2n}$ since

$$Lim_{j \in N} (a^j_i) = a^*_i > 0$$

and

$$Lim_{j \in N} (b^*_i) = b^*_i > 0.$$

Now, it is easy to compute that any agglomeration point (x,y) of the sequence (x^j, y^j) verifies that (x,y) is feasible in (1) and

$$Lim_{j \in N} (\boldsymbol{m}^j) = = -x^*_1 a^*_1 - x^*_2 a^*_2 \ldots - x^*_n a^*_n + y^*_1 b^*_1 + y^*_2 b^*_2 + \ldots + y^*_n b^*_n.$$

Consequently,

$$Lim_{j \in N} (\boldsymbol{m}^j) \leq \boldsymbol{m}.$$

Let us prove the reverse inequality. If $\boldsymbol{m} = 0$ there is nothing to prove. So, we assume that $\boldsymbol{m} > 0$. As in the first part of the proof, we denote by f^j and f the corresponding functions f with prices (b^j, a^j, B^j, A^j) and (b^*, a^*, B^*, A^*) respectively. Theorem 2 proves that there exists (x^*, y^*) verifying

$$x_1 B^*_1(k) + x_2 B^*_2(k) + \ldots + x_n B^*_n(k) - y_1 A^*_1(k) - y_2 A^*_2(k) - \ldots - y_n A^*_n(k) \geq 0,\ k \in K$$

such that $f(x^*, y^*) = \boldsymbol{m}$. Since $\boldsymbol{m} > 0$ we get that $y^* \neq 0$ and thus f is continuous in (x^*, y^*). Consequently, for a given $\varepsilon > 0$ there exists $\delta > 0$ such that $f(x', y') > \boldsymbol{m} - \varepsilon\ (>0)$ and

$$x'_1 B^*_1(k) + x'_2 B^*_2(k) + \ldots + x'_n B^*_n(k) - y'_1 A^*_1(k) - y'_2 A^*_2(k) - \ldots - y'_n A^*_n(k) > 0,$$

for every $k \in K$, where $(x', y') = (\delta + x^*_1, x^*_2, \ldots x^*_n, y^*)$. We now introduce for such an (x', y') the functions $h: \Pi \to \mathbb{R}$ and $G: \Gamma \to C(K)$ by

$$h(b,a) = (-x'_1 a_1 - x'_2 a_2 \ldots - x'_n a_n + y'_1 b_1 + y'_2 b_2 + \ldots + y'_n b_n)/(y'_1 b_1 + y'_2 b_2 + \ldots + y'_n b_n)$$
$$G(B,A) = x'_1 B_1 + x'_2 B_2 + \ldots + x'_n B_n(k) - y'_1 A_1 - y'_2 A_2 - \ldots - y'_n A_n.$$

Since h is continuous in (b^*, a^*) and G is continuous in (B^*, A^*) then there exists j_0 such that $h(b^j, a^j) > \boldsymbol{m} - \varepsilon\ (>0)$ and $G(B^j, A^j) > 0$ for $j \geq j_0$. Thus,

$$\boldsymbol{m}^j \geq f(x', y') = h(b^j, a^j) > \boldsymbol{m} - \varepsilon$$

for every $j \geq j_0$, and the inequality $\mathrm{Lim}_{j \in N}\ \boldsymbol{m}^j \geq \boldsymbol{m}$ is proved.

The Dual Approach

In this section we turn our attention to the dual problem of Problem (1). The first interest of such an approach is that it allows us to interpret the measure $\boldsymbol{m}$ as the maximum relative error in pricing each asset for a determined "state" measure. Second it provides a way to relate the measure $\boldsymbol{m}$ with the frictionless measures $\boldsymbol{m}_{p,\alpha}$ where the securities prices p, α lie between the bid and the ask prices b and a, B and A respectively. As a consequence dual problems provide a characterization of no arbitrage in presence of frictions by means of the existence of non negative linear pricing rules for an adequate frictionless model.

Denoting the dual variables by $\gamma \in \mathbb{R}$ and $\mu \in M(K)$ respectively, the dual problem of Problem (1) is[1]

[1] See, for instance, Anderson and Nash (1987), Balbás and Guerra (1996) or Balbás *et al.* (1999).

$$Minimize\ \gamma \tag{2}$$
$$\int_K B_i d\mu \le a_i, \text{ for every } i=1,2,\ldots,n,$$
$$\int_K A_i d\mu + \gamma b_i \ge b_i, \text{ for every } i=1,2,\ldots,n$$
$$\mu \in M_+(K),\ \gamma \ge 0.$$

Although the variable in Problem (1) takes values in $\mathbb{R}^{2n}$, there is an inequality constraint taking values in an infinite dimensional space *C(K)*. Thus we must prove that there is no duality gap for (1) and (2).

Lemma 4. There is strong duality for (1) (*i.e.,*(1) and (2) are both solvable and there is no duality gap for (1) and (2)).

Proof. Recall that the positive cone of *C(K)* has non empty interior and that according to our assumptions B_1 and A_1 are interior points of $C_+(K)$. Thus, the conditions of Lagrange duality theorem (see Luenberger (1969) pp. 224) hold for (1) and (2). Consequently, there is no duality gap for (1) and (2) and (2) is solvable.

In the latter theorem, the fact that *C(K)* has a positive cone with non empty interior has been used to prove the absence of duality gap and the solvability of Problem (2). However the absence of duality gap can be proved in other topological frameworks. For instance, if we assume that A_i and $B_i \in L^p(\Omega,\Sigma,P)$ with $1 \le p \le \infty$, one can prove that there is no duality gap for the corresponding problems (1) and (2). Unfortunately, if no additional assumptions are imposed, the solvability of Problem (2) cannot be stated except for $p=\infty$.

The lemma above is the key to prove a first characterization of the absence of arbitrage.

Lemma 5. The bid-ask prices model admits no arbitrage if and only if there exists a measure $\mu \in M_+(K)$, $\mu \neq 0$, such that

$$Max\ \{(\textstyle\int_K B_i d\mu)/a_i,\ i=1,2,\ldots,n\} \le Min\ \{(\textstyle\int_K A_i d\mu)/b_i,\ i=1,2,\ldots,n\}.$$

Furthermore, in the affirmative case, latter inequality holds for any $\mu \in M_+(K)$, such that

$$(\mu,,0) \in M_+(K) \times \mathbb{R},\ \mu \neq 0,$$

solves the dual problem[1].

Proof. Assume that the model is arbitrage free. Then, Theorem 1 and the latter lemma guarantee the existence of $\mu \in M_+(K)$ such that $(\mu,0) \in M_+(K) \times \mathbb{R}$ solves the dual problem. Consequently $(\mu,0)$ must be dual feasible, from where $\mu \neq 0$ and the required condition trivially follows.

Conversely, the inequality above implies the existence of $\tau \in \mathbb{R}$ lying between both terms. Clearly

$$\tau \ge (\textstyle\int KB1d\mu)/a1 > 0.$$

Therefore, it may be assumed that $\tau = 1$ since μ may be replaced by μ/τ otherwise. Thus, $(\mu,,0) \in M_+(K) \times \mathbb{R}$ is feasible for Problem (2) and $\boldsymbol{m}=0$.

We define the functions β, β_1 and β_2 from $M_+(K)$ to $\mathbb{R}$ by

$$\beta_1(\mu) = Min\ \{(\textstyle\int_K A_i d\mu)/b_i,\ i=1,2,\ldots,n\},$$
$$\beta_2(\mu) = Max\ \{(\textstyle\int_K B_i d\mu)/a_i,\ i=1,2,\ldots,n\}$$

and

$$\beta(\mu) = Max\ \{0,\ 1 - \beta_1(\mu)/\beta_2(\mu)\}$$

[1] Assume that the model is arbitrage free and consider that μ verifies the required inequality. Substituting μ by $\mu/\mu(K)$ it may be assumed that $\mu(K)=1$ and μ is a probability measure. Hence, the lemma allows us a simple interpretation. When the model is arbitrage free there exists a probability measure such that the best expected return provided by long positions is worse than the best expected return provided by short positions.

with the convention $\beta(0) = 1$.

Lemma 6. Consider the following optimization problem:

$$\text{Minimize } \beta(\mu) \qquad (3)$$
$$\mu \in M_+(K).$$

Then, Problem (3) is solvable and $\boldsymbol{m}$ is its optimal value[1].

Proof. We first prove that $\beta(\mu) \geq \boldsymbol{m}$ for any $\mu \in M_+(K)$. The inequality obviously holds for $\mu = 0$ since $\beta(0) = 1$. Assume that $\mu \neq 0$ and set $\tau = \beta_2(\mu)$ and $\nu = \tau^{-1}\mu$. Then $\beta(\mu) = \beta(\nu)$ since the function β is homogeneous, and $(\nu, \beta(\nu))) \in M_+(K)\times\mathbb{R}$ is feasible in (2). Consequently $\boldsymbol{m} \leq \beta(\nu) = \beta(\mu)$.

Finally, let $(\mu,,\boldsymbol{m}) \in M_+(K)\times\mathbb{R}$ be a solution in (2). In particular from the feasibility of $(\mu,,\boldsymbol{m})$ in (2) we get that $\beta_2(\mu) \leq 1$ and $\boldsymbol{m} \geq Max\ \{0, 1\text{-}\beta_1(\mu)\}$. Combining both inequalities we obtain that $\boldsymbol{m} \geq \beta(\mu)$ and using the first part of the proof we conclude that $\boldsymbol{m} = \beta(\mu)$.

We are now in a position to prove our main result concerning the relation of measuring with bid-ask prices and measuring with frictionless prices. We find that for the measure $\boldsymbol{m}$ there exist prices p and α, lying between the bid and the ask prices, such that $\boldsymbol{m}$ is the maximum relative profit available in the frictionless market with prices p and α. Moreover, each (p_i,α_i) can be chosen to be a point in the line segment joining (b_i,A_i) and (a_i,B_i).This property significantly simplifies the computation of (p,α) in practical situations.

Theorem 7. The following conditions hold:

i) $\boldsymbol{m} = Min\ \{\ \boldsymbol{m}_{p,\alpha} : b_i \leq p_i \leq a_i,\ B_i \leq \alpha_i \leq A_i,\ i=1,2,...,n\}$.

ii) Moreover, for every $i=1,2,...,n$ there exists r_i in $[0,1]$ such that $\boldsymbol{m} = \boldsymbol{m}_{p',\alpha'}$ with

$$p'_i = (1\text{-}r_i)a_i + r_i b_i \text{ and } \alpha'_i = (1\text{-}r_i)B_i + r_i A_i\,.$$

Proof. We first prove that $\boldsymbol{m} \leq \boldsymbol{m}_{p,\alpha}$. Let (p,α) be such that $b \leq p \leq a$ and $B \leq \alpha \leq A$ and denote by $(3_{p,\alpha})$ the corresponding frictionless Problem (3) with prices (p,α), *i.e.*,

$$\text{Minimize } \beta_{p,\alpha}(\mu)$$
$$\mu \in M_+(K),$$

where $\beta_{p,\alpha}(0) = 1$ and for $\mu \neq 0$ we define $\beta^1_{p,\alpha}(\mu) = \text{Min}\ \{(\int_K \alpha_i d\mu)/p_i\text{: } i=1,2,\dots,n\}$, $\beta^2_{p,\alpha}(\mu) = \text{Max}\ \{(\int_K \alpha_i d\mu)/p_i\text{: } i=1,2,\dots,n\}$ and $\beta_{p,\alpha}(\mu)= 1\text{-}(\beta^1_{p,\alpha}(\mu)/\beta^2_{p,\alpha}(\mu))$. By Lemma 6, Problems (3) and $(3_{p,\alpha})$ achieve their respectively optimal value $\mathbf{m}$ and $\mathbf{m}_{p,\alpha}$. If $\mathbf{m}_{p,\alpha} = 1$ then $\mathbf{m} \leq \mathbf{m}_{p,\alpha}$. Otherwise, there exists $\mu \in M_+(K)$ such that $\mathbf{m}_{p,\alpha} = 1 - \beta_{p,\alpha}(\mu)$. Thus,

$$0 < \beta^1_{p,\alpha}(\mu) \leq \beta_1(\mu)$$

and

$$\beta^2_{p,\alpha}(\mu) \geq \beta_2(\mu).$$

Consequently $\mathbf{m} \leq \beta(\mu) \leq \mathbf{m}_{p,\alpha}$.

Let us prove that condition *ii)* holds (thus condition *i)* also holds). Assume first that $\boldsymbol{m} > 0$ and let $\mu \in M_+(K)$ such that $\boldsymbol{m} = \beta(\mu)$. From Lemma 5 we know that $\beta_1(\mu) < \beta_2(\mu)$. Let I and J be two indexes with $\beta_1(\mu)=(\int_K A_I d\mu)/b_I$ and $\beta_2(\mu)=(\int_K B_J d\mu)/a_J$. Now we proceed to set p' and α'.

If i=I take $\alpha'_I = A_I$ and $p'_I = b_I$..

If i=J take $\alpha'_I = B_I$ and $p'_I = a_I$..

Whenever $i \neq I$ and $i \neq J$, set $v_i=(\int_K A_i d\mu)/b_i$ and $u_i=(\int_K B_i d\mu)/a_i$.

[1] This lemma yields new interpretations. So, a probability measure μ solving (3) may be considered as a proxy of state prices when the model is not arbitrage free, and $\boldsymbol{m} = \beta(\mu)$ may be understood as the error of the investors when they price the securities and compare returns corresponding to long and short positions.

If $u_i \le \beta_1(\mu) \le v_i \le \beta_2(\mu)$ take $\alpha'_i = A_i$ and $p'_i = b_i$.
If $\beta_1(\mu) \le u_i \le \beta_2(\mu) \le v_i$ take $\alpha'_i = B_i$ and $p'_i = a_i$.

If $\beta_1\mu) \le u_i \le v_i \le \beta_2(\mu)$ then any choice of α'_i and p'_i as in the statement *ii)* is adequate.
Finally, if $u_i \le \beta_1(\mu) < \beta_2(\mu) \le v_i$ define the function L_i from $[0,1]$ to $\mathbb{R}$

$$L_i(r) = [\int_K (rA_i + (1-r)B_i)d\mu]/(rb_i + (1-r)a_i).$$

Since $L_i(r) = u_i$, $L_i(1) = v_i$ and L_i is continuous, it follows that there exists $r_i \in (0,1)$ such that $\beta_1(\mu) < L_i(r_i) < \beta_2(\mu)$. Take $\alpha'_i = (1-r_i)B_i + r_iA_i$ and $p'_i = (1-r_i)a_i + r_ib_i$. It is straightforward to verify that $\boldsymbol{m} = \boldsymbol{m}_{p',\alpha'}$.

It only remains to prove *ii)* whenever $\boldsymbol{m} = 0$. From Lemma 5 there exists $\mu \in M_+(K)$, $\mu \neq 0$, such that $\beta_1(\mu) \ge \beta_2(\mu)$, with the same notations as above. Fix a point $s \in [\beta_2(\mu), \beta_1(\mu)]$. Since for every i we have that $u_i \le \beta_2(\mu) \le \beta_1(\mu) \le v_i$, proceeding as above we get r_i in $[0,1]$ such that $L_i(r) = s$, $i=1,2,\ldots,n$. Thus, if $\alpha'_i = (1-r_i)B_i + r_iA_i$ and $p'_i = (1-r_i)a_i + r_ib_i$ we have that $\boldsymbol{m}_{p',\alpha'} = 0$.

As a consequence we get a characterization of the no arbitrage condition in a model with transaction costs by means of linear pricing rules.

Corollary 8. The bid-ask prices model admits no arbitrage if and only if there exist at least a measure $\mu \in M_+(K)$ and $p \in \mathbb{R}^n$, $\alpha \in C(K)^n$ satisfying $b \le p \le a$, $B \le \alpha \le A$ such that $\int_K \alpha_i d\mu = p_i$ for every $i=1,2,\ldots,n$. Furthermore, in the affirmative case, p and α may be chosen in such a way that for every $i=1,2,\ldots,n$ there exists r_i in [0,1] with

$$p_i = (1-r_i)a_i + r_ib_i \text{ and } \alpha'_i = (1-r_i)B_i + r_iA_i .$$

Proof. The bid-ask prices model is arbitrage free if and only if there exist $p \in \mathbb{R}^n$ and $\alpha \in C(K)^n$ such that $b \le p \le a$, $B \le \alpha \le A$ and $\boldsymbol{m}_{p,\alpha} = 0$, and therefore, Lemma 5 (applied for the frictionless case) guarantees that $\boldsymbol{m}_{p,\alpha} = 0$ if and only if there exists $\mu \in M_+(K)$ such that $\int_K \alpha_i d\mu = p_i$ for every $i=1,2,\ldots,n$.

Conclusions

A measurement of the arbitrage opportunities has been developed for a model with transaction costs. This is important since previous literature usually focuses on the perfect market case to analyze the level of cross-market arbitrage and integration of several financial markets, and this makes it difficult to precise if the arbitrage existence still holds after discounting market imperfections.

The measure allows us to discount several sorts of imperfections. For instance, imperfections due to the bid-ask spread and imperfections that depend on the total traded value.

The measure may be easily computed in empirical studies and seems to be a practical tool when testing the level of integration between financial markets, analyzing the existence of arbitrage portfolios in real markets or pricing and hedging some derivative securities, amongst other possibilities.

The measure is continuous with respect to the initial parameters and data, and may be introduced by a primal and a dual optimization problem. The primal problem permits us to interpret the measure in terms of relative arbitrage gains, while the dual one indicates the error committed by the investors when they price the securities. Besides, the dual problem yields a proxy of state prices in the imperfect case and shows that the measure coincides with the minimum measure of arbitrage associated to frictionless models for which prices and payoffs are lying within the spreads and are given by convex combinations of bid and ask prices and payoffs. As a consequence, the arbitrage absence is also related to the existence of associated arbitrage free frictionless markets.

Finally, the theory may be adapted so that it can apply in two different problems: the measurement of the degree of fulfilment in practice of theoretical asset pricing models, and the existence of arbitrage and a term structure of interest rates in imperfect bonds markets.

References

1. Anderson, E.J. and P. Nash (1987). "Linear Programming in Infinite-Dimensional Spaces". John Wiley & Sons, New York.
2. Balbás, A. and P.J. Guerra (1996). "Sensitivity Analysis for Convex Multiobjective Programming in Abstract Spaces". Journal of Mathematical Analysis and Applications, 202, 645-658.
3. Balbás, A., P.J. Guerra and A. Heras (1999). "Duality Theory and Slackness Conditions in Multiobjective Linear Programming". Computers & Mathematics with Applications, 37, 101-110.
4. Balbás, A., I.R. Longarela and J. Lucia (1999). "How Financial Theory Applies to Catastrophe-Linked Derivatives: An Empirical Test of Several Pricing Models". Journal of Risk and Insurance, 66, 4, 551-582.
5. Balbás, A., I.R. Longarela and A. Pardo (2000). "Integration and Arbitrage in the Spanish Financial Markets: An Empirical Approach". Journal of Futures Markets, 20, 4, 321-344.
6. Balbás, A. and Muñoz (1998). "Measuring the Degree of Fulfillment of the Law of One Price. Applications to Financial Market Integration". Investigaciones Económicas, 22 , 2, 153-177.
7. Chamberlain, G. and M. Rothschild (1983). "Arbitrage, Factor Structure, and Mean-Variance Analysis on Large Assets". Econometrica, 51, 1281-1304.
8. Chen, Z. and P.J. Knez (1995). "Measurement of Market Integration and Arbitrage". Review of Financial Studies, 8, 2, 545-560.
9. Hansen, L.P. and R. Jagannathan (1997). "Assessing Specification Errors in Stochastic Discount Factor Models". The Journal of Finance, 52, 2, 567-590.
10. Harrison, J. and D. Kreps (1979). "Martingales and Arbitrage in Multiperiod Securities Markets'". Journal of Economic Theory, 20, 381-408.
11. Ingersoll, J. (1989). "Theory of Financial Decision Making". Totawa, NJ: Rowan Littlefield.
12. Jaschke, S.R. (1998). "Arbitrage Bounds for the Term Structure of Interest Rates". Finance and Stochastics, 2, 29-40.
13. Jouini, E. and H. Kallal (1995). "Martingales and Arbitrage in Securities Markets with Transaction Costs". Journal of Economic Theory, 66, 178-197.
14. Kamara, A. and T.W. Miller, Jr (1995)." Daily and Intradaily Tests of European Put-Call Parity". Journal of Financial and Quantitative Analysis, 30, 4, 519-541.
15. Kempf, A. and O. Korn (1998). "Trading System and Market Integration". Journal of Financial Intermediation, 7, 220-239.
16. Kleidon, A.W. and R.E. Whaley (1992). "One Market? Stocks, Futures and Options during October 1987". The Journal of Finance, 67, 3, 851-877.
17. Luenberger, D.G. (1969). "Optimization by Vector Space Methods". John Wiley, New York.
18. Pham, H. and N. Touzi (1999). "The Fundamental Theorem of Asset Pricing with Cone Constraints". Journal of Mathematical Economics, 31, 265-279.