CONSISTENCY OF (INTERTEMPORAL) BETA ASSET PRICING AND BLACK-SCHOLES OPTION VALUATION

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Abstract

It is well-known that the CAPM valuation formula results from a quadratic utility of the representative investor. In this paper we show that the CAPM valuation rule remains valid if the representative investor exhibits an exponential utility and asset and market returns are bivariate normally distributed. In contrast to quadratic utility, exponential utility implies a positive stochastic discount factor that guarantees positive (option) prices. In particular, within our discrete-time framework, options are priced according to the Black-Scholes formula.

In addition, our approach allows the valuation of single assets if their returns follow an intertemporal market model with stochastic beta. The resulting valuation formula differs from the standard CAPM only in that the expected beta replaces the deterministic one. It turns out that the expected beta can easily be estimated from the return time series.

Key words: Asset pricing, option valuation, intertemporal CAPM.
JEL Classification: G12, G13.

1. Introduction

The capital asset pricing model (CAPM) is still regarded as a paradigm of capital market theory[1]. Primarily, this may be due to its simple structure. Besides, during the last decades, research was more concentrated on the pricing of derivative instruments rather than on the valuation of underlying assets.

In its standard version the CAPM claims that the expected return of a risky asset consists of two parts. The first one is a liquidity premium at the level of the risk-free interest rate. The second one is a risk premium that equals the market risk premium adjusted for the systematic risk of the asset.

Thus, the attractiveness of the model comes from two facts. On one hand, the linearity of the valuation formula is consistent with an arbitrage-free capital market because a portfolio's beta equals the weighted average of the betas of the stocks that constitute the portfolio. On the other hand, the model in its standard version has the economically comprehensible interpretation that only the systematic risk factor is valued because unsystematic risks can be diversified.

Although there is a large number of special versions[2] and versions with weaker assumptions[3] the model is exposed to frequent critique. Fama and French (1992) found out that beta coefficients even in long time series do not have any explanatory power to the cross-section of returns at the stock market. Instead, differences within this cross-section are explained by microeconomic factors like the market value and the book-to-market ratio of equity[4]. Furthermore, Roll and Ross (1994) replied on the basis of Roll's (1977) critique. They showed that a small degree of ineffi-

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[1] Two main applications of the CAPM can be found in performance measurement and company valuation. For example, auditors suggest to compute the cost of equity via the so called tax CAPM of Brennan (1970).

[2] Only few of the variety of examples are the tax CAPM of Brennan (1970) mentioned above, the intertemporal CAPM of Merton (1973a), the multi-beta CAPM of Losq and Chateau (1982), the consumption-based CAPM and the lognormal CAPM of Rubinstein (1976), and the CAPM with stochastic inflation of Roll (1973).

[3] One example is Black's (1972) zero-beta version without a risk-free asset. Turnbull (1977) analyzed special versions with respect to market imperfections. Lintner (1969) showed that heterogeneous expectations do not lead to severe deviations in asset valuation.

[4] Former studies could rather confirm the CAPM. (See Black, Jensen and Scholes, 1972; and Fama and Macbeth, 1973). However, at the latest since the (Fama and French, 1992) study, the standard CAPM is regarded as empirically falsified.

ciency of the market index, that is used to compute the beta coefficients, is sufficient to observe a covariance of zero between beta coefficient and mean return[1].

Since the standard CAPM is regarded as empirically falsified, we turn to the following theoretical point of critique. Dybvig and Ingersoll (1982) showed that the valuation formula of the CAPM results from a linear stochastic discount factor used by the representative investor to value risky future cash flows. The stochastic discount factor represents the marginal utility of a representative investor. A linear marginal utility results from a quadratic utility function. Hence, the quadratic utility function implies the CAPM valuation rule. But a quadratic utility function exhibits a negative marginal utility in the end and increasing relative and absolute risk aversion. Furthermore, negative (option) prices may occur if the representative investor exhibits a quadratic utility.

In this paper we show that the valuation formula of the CAPM likewise results from a representative investor with an exponential utility function in case of bivariate normally distributed asset and market returns. In contrast to a quadratic utility, an exponential utility function always exhibits a positive marginal utility and constant absolute risk aversion. Furthermore, mean and variance completely characterize normally distributed random variables. With normally distributed asset returns, in market equilibrium the CAPM valuation rule can be derived from mean-variance portfolio selection. Furthermore, it turns out that in our discrete-time model with exponential utility and normally distributed asset returns, options can be valued by the Black and Scholes (1973) formula. In contrast to a quadratic utility funtion, this excludes negative option prices.

Additionally, the valuation framework with exponential utility and normally distributed asset returns has the following advantage. Using the corresponding stochastic discount factor, the valuation of single financial assets is possible even in case of stochastic beta coefficients. The assumption of the market model with a stochastic beta as return generating process is motivated by Merton's (1973a) intertemporal CAPM.

In case the beta coefficient is stochastic, it is unknown at each point in time. In this situation the least squares regression analysis is not helpful because it yields just one single estimator. The valuation by a representative investor with exponential utility results in a valuation formula that is linear in the expected beta coefficient if asset betas and the market return are bivariate normally distributed. Fortunately, it is possible to estimate the expected beta from the return time series without the need for observing realized beta coefficients. On this note, we present a testable version of Merton's (1973a) intertemporal CAPM.

The paper is organized as follows. In section 2 the valuation of options and underlying assets by a representative investor is presented. The representative investor exhibits a quadratic utility function at first and an exponential utility function afterwards. Section 3 deals with stochastic beta coefficients. The valuation formula with expected beta is derived. Section 4 concludes with a brief summary.

2. Valuation Framework with a Representative Investor

Let P_0 denote the current price and P the random future price of an investment at the end of the period in our one-period model. The return R on a portfolio consisting of this investment with fraction x and the risk-free asset with interest rate r_f with fraction $(1-x)$ reads as follows:

$$R = x \cdot R_P + (1-x) \cdot r_f \quad \text{where} \quad R_P \equiv \frac{P - P_0}{P_0}. \tag{1}$$

The maximization of expected utility with respect to the fraction x gives the following first order condition:

[1] If the security market line is used in performance measurement, nearly every ranking can be created by an appropriate index choice (See Dybvig and Ross, 1985).

$$\frac{\partial \mathrm{E}(u(R))}{\partial x}\Big|_{x^*} = \mathrm{E}\left(\frac{\partial u(R)}{\partial R}\Big|_{x^*} \cdot \underbrace{(R_P - r_f)}_{=\frac{P}{P_0}-(1+r_f)}\right) \overset{!}{=} 0. \tag{2}$$

This leads to the following valuation P_0 of the risky future cash flow P:

$$P_0 = \frac{1}{1+r_f} \cdot \mathrm{E}\left(\frac{\partial u(R)/\partial R}{\mathrm{E}(\partial u(R)/\partial R)}\Big|_{x^*} \cdot P\right). \tag{3}$$

The risky future cash flow is transformed in two steps. At first, it is transformed into a corresponding risk-free cash flow that is discounted by the risk-free interest rate subsequently. Thus, the stochastic discount factor

$$\omega \equiv \frac{\partial u(R)/\partial R}{\mathrm{E}(\partial u(R)/\partial R)}\Big|_{x^*} \tag{4}$$

gives a characterization of the marginal rate of substitution between a risky and a risk-free cash flow[1]. The valuation according to equation (3) corresponds to the risk-neutral valuation technique. With the state-price density ω, the mean $\mathrm{E}(\omega \cdot P)$ is discounted by the risk-free interest rate. In our discrete-time single-period model, the incompleteness of the market with continuous states is expressed by the dependency of the valuation on risk preferences.

The stochastic discount factor ω represents the normalized marginal utility of the representative investor. Therefore, its mean equals unity:

$$\mathrm{E}(\omega) = 1. \tag{5}$$

By rearranging equation (2), it can be seen that the risk-neutral valuation of the risky return R_P equals the risk-free interest rate:

$$r_f = \mathrm{E}(\omega \cdot R_P). \tag{6}$$

2.1. Representative Investor with a Quadratic Utility Function

Dybvig and Ingersoll (1982) showed that the CAPM valuation formula holds if the representative investor exhibits a quadratic utility function. However, in this case negative option prices may occur in an incomplete market with discrete-time trading. A derivation of the (Dybvig and Ingersoll, 1982) result is given in the following.

Let $r \equiv R - r_f$ denote the excess return. For efficiency reasons we analyze portfolios which consist of the market portfolio M with fraction x and the risk-free asset with fraction $(1-x)$. Moreover, we use the following quadratic utility function:

$$u(R) = -\frac{1}{2} \cdot R^2 + b \cdot R + c \quad \text{where} \quad R = x \cdot R_M + (1-x) \cdot r_f = x \cdot r_M + r_f. \tag{7}$$

The maximization of expected utility with respect to the fraction of the market portfolio gives the following first order condition:

$$\frac{\partial \mathrm{E}(u(R))}{\partial x}\Big|_{x^*} = \mathrm{E}((-R|_{x^*} + b) \cdot r_M) = \mathrm{E}(-(x^* \cdot r_M + r_f - b) \cdot r_M) \overset{!}{=} 0. \tag{8}$$

Rearranging yields the optimal fraction of the market portfolio:

[1] See also Glosten and Jagannathan (1994).

$$x^* = -\frac{(r_f - b)\cdot \mathrm{E}(r_M)}{\mathrm{E}(r_M^2)}. \tag{9}$$

To compute the stochastic discount factor of a quadratic utility function, we put this fraction into the derivative of the utility function $u(R)$ with respect to R:

$$\frac{\partial u(R)}{\partial R}\Big|_{x^*} = -R\big|_{x^*} + b = \frac{(b - r_f)\cdot(\mathrm{E}(r_M^2) - \mathrm{E}(r_M)\cdot r_M)}{\mathrm{E}(r_M^2)}. \tag{10}$$

Taking the mean yields:

$$\mathrm{E}\left(\frac{\partial u(R)}{\partial R}\Big|_{x^*}\right) = \frac{(b - r_f)\cdot\left(\mathrm{E}(r_M^2) - \mathrm{E}^2(r_M)\right)}{\mathrm{E}(r_M^2)}. \tag{11}$$

Finally, using market volatility σ_M, the stochastic discount factor ω^{qu} of a quadratic utility function reads as follows:

$$\omega^{qu} = \frac{\mathrm{E}(r_M^2) - \mathrm{E}(r_M)\cdot r_M}{\sigma_M^2} = 1 - \frac{\mathrm{E}(r_M)}{\sigma_M^2}\cdot(r_M - \mathrm{E}(r_M)). \tag{12}$$

To show that the valuation rule with the quadratic stochastic discount factor leads to the CAPM equation, we use equation (6) and get:

$$r_f = \mathrm{E}(\omega^{qu}\cdot R_P) = \mathrm{E}\left(R_P - \frac{\mathrm{E}(R_M) - r_f}{\sigma_M^2}\cdot(R_M - \mathrm{E}(R_M))\cdot R_P\right). \tag{13}$$

Rearranging leads to the CAPM valuation rule:

$$\mathrm{E}(R_P) = r_f + \underbrace{\frac{\mathrm{Cov}(R_P, R_M)}{\sigma_M^2}}_{\equiv\beta_P}\cdot(\mathrm{E}(R_M) - r_f). \tag{14}$$

Equation (12) shows that the quadratic stochastic discount factor is linear in the return of the market portfolio with a negative slope. If the end-of-period price and the return of the market portfolio, respectively, are unbounded, the quadratic stochastic discount factor can take negative values. According to equation (12), this occurs if

$$r_M > \frac{\sigma_M^2}{\mathrm{E}(r_M)} + \mathrm{E}(r_M) = \frac{\mathrm{E}(r_M^2)}{\mathrm{E}(r_M)}. \tag{15}$$

A negative stochastic discount factor may lead to negative asset prices. Examples are call options on the market index that are deep out-of-the-money. If the payoff of such a call is positive only if condition (15) is fulfilled, its payoff is positive only if the valuation factor takes negative values. Therefore, call options on the market index that are sufficiently deep out-of-the-money get a negative price. Of course, this does not contradict the law of one price. However, an arbitrage-free market asks for a positive stochastic discount factor.

Jarrow and Madan (1997) concluded the linearity of the stochastic discount factor from the mean-variance criterion. However, this implication only holds for arbitrarily distributed returns that are continuously computed. Otherwise, i.e. without any assumption concerning the return distribution, only the quadratic utility function leads to a mean-variance criterion. Conversely, the following section shows that with bivariate normally distributed asset and market returns an exponential stochastic discount factor implies the valuation formula of the CAPM, too.

2.2. Representative Investor with an Exponential Utility Function

To show how the CAPM valuation rule is related to a representative investor with an exponential utility function, we assume:

(A1) The representative investor exhibits constant absolute risk aversion, i.e. an exponential utility function.

(A2) Returns of single assets and the market return are bivariate normally distributed.

Using these assumptions we obtain the following proposition:

Proposition 1: Under the assumptions (A1) and (A2), the CAPM valuation formula holds.

Proof: With constant absolute risk aversion a, i.e. exponential utility function

$$u(R) = -\exp\{-a \cdot R\} \quad \text{where} \quad R = x \cdot r_M + r_f, \tag{16}$$

and with normally distributed return R, it is sufficient to maximize the certainty equivalent

$$\mathrm{CE}(R(x)) = \mathrm{E}(R(x)) - \frac{a}{2} \cdot \mathrm{Var}(R(x)) = x \cdot \mathrm{E}(r_M) + r_f - \frac{a}{2} \cdot x^2 \cdot \sigma_M^2. \tag{17}$$

The maximization of the certainty equivalent CE(R(x)) with respect to the fraction of the market portfolio x yields:

$$x^* = \frac{\mathrm{E}(r_M)}{a \cdot \sigma_M^2}. \tag{18}$$

With the optimal fraction x^* of the market portfolio, we are able to derive the exponential stochastic discount factor. The derivation of the utility function $u(R)$ with respect to R reads:

$$\frac{\partial u(R)}{\partial R}\bigg|_{x^*} = a \cdot \exp\{-a \cdot r_f\} \cdot \exp\left\{-\frac{\mathrm{E}(r_M)}{\sigma_M^2} \cdot r_M\right\}. \tag{19}$$

Taking the mean yields:[1]

$$\mathrm{E}\left(\frac{\partial u(R)}{\partial R}\right)\bigg|_{x^*} = a \cdot \exp\{-a \cdot r_f\} \cdot \exp\left\{-\frac{\mathrm{E}^2(r_M)}{2 \cdot \sigma_M^2}\right\}. \tag{20}$$

Finally, the stochastic discount factor $\omega^{\exp}$ of an exponential utility function reads as follows:

$$\omega^{\exp} = \exp\left\{-\frac{\mathrm{E}(r_M)}{\sigma_M^2} \cdot \left(r_M - \frac{\mathrm{E}(r_M)}{2}\right)\right\}. \tag{21}$$

To show that the valuation rule with the exponential stochastic discount factor leads to the CAPM equation, we again use equation (6) and get:

$$r_f = \mathrm{E}(\omega^{\exp} \cdot R_P) = \mathrm{E}\left(\exp\left\{-\frac{\mathrm{E}(r_M)}{\sigma_M^2} \cdot \left(r_M - \frac{\mathrm{E}(r_M)}{2}\right)\right\} \cdot R_P\right). \tag{22}$$

If asset and market returns have a bivariate normal distribution the Stein-Rubinstein covariance formula yields[2]:

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[1] Here, we utilize that for a normally distributed random variable X with mean μ and standard deviation σ it holds that $\mathrm{E}(\exp\{X\}) = \exp\left\{\mu + \frac{\sigma^2}{2}\right\}$.

[2] If random variables X and Y have a bivariate normal distribution, g: $\Re \to \Re$ is an at least once continuously differentiable function, $\mathrm{E}(g'(Y))$ exists, and $\lim_{y \to \pm\infty} g(y) \cdot f(y) = 0$, where $f(y)$ is the density of Y, then

$$r_f = -\frac{E(R_M)}{\sigma_M^2} \cdot \underbrace{E(\omega^{\exp})}_{=1} \cdot \text{Cov}(r_M, R_P) + \underbrace{E(w^{\exp})}_{=1} \cdot E(R_P). \tag{23}$$

Rearranging and replacing r_M by (R_M–r_f) leads to the CAPM valuation formula:

$$E(R_P) = r_f + \beta_P \cdot (E(R_M) - r_f). \tag{24}$$

Hence, with a bivariate normal distribution of asset and market returns, the CAPM valuation rule can be derived under the assumption that the representative investor exhibits an exponential utility. Therefore, the CAPM does not necessarily imply the quadratic utility function of the representative investor.

To value options with the exponential stochastic discount factor, we consider the payoff C of a European index call option with exercise price K:

$$C = \max\{M - K; 0\}, \tag{25}$$

where M denotes the index price. Furthermore, we assume:

(A3) The index price M follows a geometric Brownian motion:

$$\frac{dM(t)}{M(t)} = \mu_M dt + \sigma_M \, dW(t). \tag{26}$$

Using assumption (A3) a continuous-time stochastic index price process is supposed. However, we are only interested in the end-of-period price and the return over the entire period, respectively. That means that we assume a continuous state space with discrete trading points in time. Note, that in contrast to the Black and Scholes (1973) and Merton (1973b) option valuation framework, in our discrete-time model, it is not possible to duplicate the option's payoff. Nevertheless, using assumption (A3) we formulate the following proposition that gives the relationship between a representative investor with exponential utility and the Black-Scholes option pricing formula:

Proposition 2: Under the assumptions (A1) and (A3), options are valued according to the Black-Scholes formula.

Proof: The solution of the stochastic differential equation (26) in assumption (A3) reads as follows:

$$M(t) = M(0) \cdot \exp\left\{\left(\mu_M - \frac{\sigma_M^2}{2}\right) \cdot t + \sigma_M \cdot W(t)\right\}. \tag{27}$$

Rearranging leads to:

$$\ln\frac{M(t)}{M(0)} + \frac{\sigma_M^2}{2} \cdot t = \mu_M \cdot t + \sigma_M \cdot W(t). \tag{28}$$

If we write equation (26) as a stochastic integral equation, we get:

$$\int_0^t \frac{1}{M(s)} dM(s) = \mu_M \cdot t + \sigma_M \cdot W(t). \tag{29}$$

Now, we define the index return as

$$R_M(t) \equiv \int_0^t \frac{1}{M(s)} dM(s). \tag{30}$$

Using equations (28) and (29) yields:

$\text{Cov}(X, g(Y)) = E(d\, g(Y) / d\, y) \cdot \text{Cov}(X, Y)$. A proof of the Stein-Rubinstein covariance formula can be found in Rubinstein (1976).

$$R_M(t) = \ln\frac{M(t)}{M(0)} + \frac{\sigma_M^2}{2}\cdot t. \quad (31)$$

Hence, the expected index return equals the drift μ_M from the geometric Brownian motion (26) times time t. In our one-period model, we simplify the notation to[1]:

$$R_M = \ln\frac{M}{M_0} + \frac{\sigma_M^2}{2}. \quad (32)$$

Let r_f^c denote the continuously compounded risk-free interest rate that corresponds to the discretely compounded risk-free interest rate r_f in equation (3). Then, the valuation of the index call option using the exponential stochastic discount factor according to assumption (A1) reads as follows:

$$\begin{aligned}
C_0 &= e^{-r_f^c}\cdot \mathrm{E}(\omega^{\exp}\cdot C)\\
&= e^{-r_f^c}\cdot\int_K^\infty \exp\left\{-\frac{\mu_M - r_f^c}{\sigma_M^2}\cdot\left(\ln\frac{M}{M_0}+\frac{\sigma_M^2}{2}-r_f^c-\frac{\mu_M - r_f^c}{2}\right)\right\}\\
&\quad\cdot (M-K)\cdot\frac{1}{\sqrt{2\cdot\pi}\cdot\sigma_M\cdot M}\cdot\exp\left\{-\left(\ln\frac{M}{M_0}-\left(\mu_M-\frac{\sigma_M^2}{2}\right)\right)^2\Big/2\cdot\sigma_M^2\right\}dM\\
&= e^{-r_f^c}\cdot\int_K^\infty\frac{M-K}{M}\cdot\frac{1}{\sqrt{2\cdot\pi}\cdot\sigma_M}\cdot\exp\left\{-\left(\ln\frac{M}{M_0}-r_f^c+\frac{\sigma_M^2}{2}\right)^2\Big/2\cdot\sigma_M^2\right\}dM\\
&= M_0\cdot\int_K^\infty\frac{1}{\sqrt{2\cdot\pi}\cdot\sigma_M\cdot M}\cdot\exp\left\{-\left(\ln\frac{M_0}{M}+r_f^c+\frac{\sigma_M^2}{2}\right)^2\Big/2\cdot\sigma_M^2\right\}dM\\
&\quad - K\cdot e^{-r_f^c}\cdot\int_K^\infty\frac{1}{\sqrt{2\cdot\pi}\cdot\sigma_M\cdot M}\cdot\exp\left\{-\left(\ln\frac{M_0}{M}+r_f^c-\frac{\sigma_M^2}{2}\right)^2\Big/2\cdot\sigma_M^2\right\}dM\\
&= M_0\cdot\int_{-\infty}^{\frac{\ln\frac{M_0}{K}+r_f^c+\frac{\sigma_M^2}{2}}{\sigma_M}}\frac{1}{\sqrt{2\cdot\pi}}\cdot\exp\left\{-\frac{y^2}{2}\right\}dy\\
&\quad - e^{-r_f^c}\cdot K\cdot\int_{-\infty}^{\frac{\ln\frac{M_0}{K}+r_f^c-\frac{\sigma_M^2}{2}}{\sigma_M}}\frac{1}{\sqrt{2\cdot\pi}}\cdot\exp\left\{-\frac{y^2}{2}\right\}dy\\
&= M_0\cdot\mathrm{N}\left(\frac{\ln\frac{M_0}{K}+r_f^c+\frac{\sigma_M^2}{2}}{\sigma_M}\right) - e^{-r_f^c}\cdot K\cdot\mathrm{N}\left(\frac{\ln\frac{M_0}{K}+r_f^c-\frac{\sigma_M^2}{2}}{\sigma_M}\right)
\end{aligned} \quad (33)$$

where $\mathrm{N}(\cdot)$ denotes the cumulative standard normal distribution function.

If the utility function of the representative investor is exponential, his marginal utility is always positive. Thus, the stochastic discount factor $\omega^{\exp}$ is always positive. In contrast to the quadratic stochastic discount factor ω^{qu}, this implies positive (option) prices.

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[1] It holds that $\mathrm{E}(M(t)) = M(0)\cdot \mathrm{e}^{\mathrm{E}(R_M(t))}$ (See also Korn and Korn, 2001).

The valuation of index options by a representative investor with an exponential utility function leads to the Black-Scholes formula if the return of the underlying asset is normally distributed. Therefore, with lognormally distributed prices of the underlying asset, the valuation of options by the Black-Scholes formula is adequate despite discrete-time trading if the representative investor exhibits constant absolute risk aversion. This completes the results of Rubinstein (1976) and Brennan (1979).

3. Valuation with Stochastic Beta Coefficients

Based on Merton's (1973a) intertemporal CAPM, we assume a return generating process that differs from the standard market model by time-dependent beta coefficients:

(A1) Asset returns follow the intertemporal market model:

$$\begin{aligned} & r_{Pt} = \beta_{Pt} \cdot r_{Mt} + \varepsilon_{Pt}, \\ \text{where} \quad & \mathrm{E}(\varepsilon_P) = 0 \quad \text{for all portfolios } P, \\ & \mathrm{Cov}(\varepsilon_{P_1}, \varepsilon_{P_2}) = 0 \quad \text{for any two different portfolios } P_1 \text{ and } P_2, \\ \text{and} \quad & \mathrm{Cov}(\varepsilon_P, r_M) = 0 \quad \text{for all portfolios } P. \end{aligned} \tag{34}$$

Additionally, we assume:

(A2) Beta coefficients and the market return are bivariate normally distributed.

Using these assumptions and assumption (A1), we obtain the following proposition that gives the relationship between a representative investor with an exponential utility function and the CAPM valuation rule in a situation with stochastic beta coefficients:

Proposition 3: With the intertemporal market model according to assumption (A4) in conjunction with assumptions (A1) and (A5), the CAPM valuation rule reads:

$$\mathrm{E}(R_P) = r_f + \mathrm{E}(\beta_P) \cdot (\mathrm{E}(R_M) - r_f). \tag{35}$$

Proof: The risk-neutral valuation of a risky return R_P using the exponential stochastic discount factor leads to the following equation:

$$r_f = \mathrm{E}(\omega^{\exp} \cdot R_P) = \mathrm{Cov}(\omega^{\exp}, R_P) + \underbrace{\mathrm{E}(\omega^{\exp})}_{=1} \cdot \mathrm{E}(R_P). \tag{36}$$

Because of $\mathrm{Cov}(\varepsilon_P, r_M) = 0$ it holds:

$$\begin{aligned} \mathrm{Cov}(\omega^{\exp}, R_P) &= \mathrm{Cov}(\omega^{\exp}, \beta_P \cdot r_M) \\ &= \mathrm{E}(\omega^{\exp} \cdot \beta_P \cdot r_M) - \underbrace{\mathrm{E}(\omega^{\exp})}_{=1} \cdot \mathrm{E}(\beta_P \cdot r_M) \\ &= \mathrm{Cov}(\omega^{\exp} \cdot r_M, \beta_P) + \mathrm{E}(\omega^{\exp} \cdot r_M) \cdot \mathrm{E}(\beta_P) - \mathrm{E}(\beta_P \cdot r_M). \end{aligned} \tag{37}$$

With the stochastic discount factor, the market return gets a valuation amounting to the risk-free interest rate. Therefore, the value of the excess return in formula (37) is zero:

$$\mathrm{E}(\omega^{\exp} \cdot r_M) = 0. \tag{38}$$

The Stein-Rubinstein covariance formula gives:

$$\begin{aligned} \mathrm{Cov}(\omega^{\exp} \cdot r_M, \beta_P) &= \mathrm{E}\left(\omega^{\exp} \cdot \left(-\frac{\mathrm{E}(r_M)}{\sigma_M^2}\right) \cdot r_M + \omega^{\exp}\right) \cdot \mathrm{Cov}(\beta_P, r_M) \\ &= \left(-\frac{\mathrm{E}(r_M)}{\sigma_M^2} \cdot \underbrace{\mathrm{E}(\omega^{\exp} \cdot r_M)}_{=0} + \underbrace{\mathrm{E}(\omega^{\exp})}_{=1}\right) \cdot \mathrm{Cov}(\beta_P, r_M) \\ &= \mathrm{Cov}(\beta_P, r_M). \end{aligned} \tag{39}$$

Putting the results of equations (38) and (39) into equation (37) and the result of equation (37) into equation (36), we get the following valuation formula:

$$r_f = \mathrm{Cov}(\beta_P, r_M) - \mathrm{E}(\beta_P \cdot r_M) + \mathrm{E}(R_P) \tag{40}$$

This yields:

$$\mathrm{E}(R_P) = r_f + \mathrm{E}(\beta_P) \cdot (\mathrm{E}(R_M) - r_f). \tag{41}$$

Hence, if the representative investor exhibits an exponential utility function and beta coefficients and the market return are bivariate normally distributed, the CAPM valuation rule with stochastic beta coefficient differs from the standard CAPM formula only in that the former reveals a linear dependency on the expected beta coefficient.

However, the beta coefficients are not known at each point in time. Therefore, it is not possible to compute their average to estimate the expected beta coefficient. Nevertheless, the following proposition states that the mean beta can be estimated from the return time series under the same assumptions which were used to derive the CAPM valuation rule in case of stochastic beta coefficients:

Proposition 4: Under the assumptions (A1), (A4), and (A5), expected beta coefficients can be estimated from the return time series.

Proof: From $\mathrm{E}(\varepsilon_P) = 0$ in equation (34), it follows that:

$$\mathrm{E}(r_P) = \mathrm{E}(\beta_P \cdot r_M) = \mathrm{Cov}(\beta_P, r_M) + \mathrm{E}(\beta_P) \cdot \mathrm{E}(r_M). \tag{42}$$

Furthermore, from $\mathrm{E}(\varepsilon_P) = 0$ and $\mathrm{Cov}(\varepsilon_P, r_M) = 0$ in equation (34), it follows that:

$$\begin{aligned} \mathrm{E}(\omega^{\exp} \cdot r_P) &= \mathrm{E}(\omega^{\exp} \cdot \beta_P \cdot r_M) \\ &= \mathrm{Cov}(\omega^{\exp} \cdot r_M, \beta_P) + \underbrace{\mathrm{E}(\omega^{\exp} \cdot r_M)}_{=0} \cdot \mathrm{E}(\beta_P) \\ &= \mathrm{Cov}(\beta_P, r_M). \end{aligned} \tag{43}$$

Putting the last result into equation (42) and rearranging yield:

$$\mathrm{E}(\beta_P) = \frac{\mathrm{E}(r_P - \mathrm{E}(\omega^{\exp} \cdot r_P)}{\mathrm{E}(r_M)}. \tag{44}$$

All expected values on the right-hand side of this equation can be estimated from the return time series.

4. Summary

The capital asset pricing model (CAPM) is regarded as empirically falsified. Furthermore, tests of the CAPM only check the efficiency of the market index used to compute the beta coefficients. Some theoretical critique applies to the fact that the valuation formula of the CAPM results from a quadratic utility function of the representative investor. A quadratic utility function implies an in-the-end-negative marginal utility and increasing risk aversion. In addition, the quadratic utility function leads to a valuation factor for risky payoffs that is linear in the market return with a negative slope. With this valuation factor, call options on the market index that are very deep out-of-the-money get negative prices.

Our framework with an exponential utility function of the representative investor under the additional assumption of normally distributed returns overcomes these weaknesses. At first, the maximization of expected utility by the representative investor leads to the CAPM valuation formula, too. However, in contrast to the quadratic stochastic discount factor, the exponential stochastic discount factor remains positive. Therefore, our approach is consistent with an arbitrage-free capital market. In particular, in our considered discrete-time model, options are priced according to the Black-Scholes formula.

Additionally, the valuation framework with an exponential utility function of the representative investor allows the valuation of single assets if their returns follow an intertemporal market model with stochastic beta coefficients. The resulting valuation formula differs from the standard CAPM rule only in that now the expected beta coefficient measures systematic risk. With the exponential stochastic discount factor, expected beta coefficients can be estimated from the return time series without the need for observing realized beta coefficients.

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