# Arbitrage and Portfolio Constraints

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## Abstract

We analyze the pricing of risky income streams in a world with competitive security markets where investors are constrained by restrictions on possible portfolio holdings. We investigate how we can transfer concepts and pricing techniques from a world without frictions to such a more realistic situation. Portfolio constraints can lead to situations where not all arbitrage opportunities are necessarily eliminated. For a world with portfolio constraints the concept of no arbitrage has to be replaced by a weaker concept which we call no unlimited arbitrage. The power of no arbitrage techniques is preserved in the sense that no specific assumptions about utility functions of investors have to be made.


**Key words:** Arbitrage, Portfolio Constraints, Asset Pricing.

## 1. Introduction

Pricing securities and risky income streams by *no arbitrage* arguments has become the cornerstone of modern asset pricing theory. No-arbitrage arguments have also been an impressive practical success. The valuation techniques derived from them have become the daily tools and workhorses of thousands of practitioners and financial engineers worldwide. The idea of no arbitrage is simple. It requires that correctly priced securities should make it impossible to achieve by financial transactions a consumption bundle at zero costs that increases some investor's utility. This idea ultimately relies on an equilibrium argument and has powerful implications for asset pricing formulas. A great deal of this power comes from the fact that the question whether or not security prices do allow arbitrage, can be inferred from observable data: the prices of actively traded securities and their payoff structure. We do not have to know the entire equilibrium. Moreover, once the "correct" security prices have been found, the price of any risky income stream which can be generated by combinations of these securities is determined. Thus security pricing by no arbitrage leads to a general valuation technique for arbitrary contingent claims, which can be generated from securities traded on financial markets.

Yet the formulas are derived under highly idealized conditions. Among them, *perfect competition* and *frictionless security trading* are the two most important ones. Evidence as well as practical experience suggest that the assumption of price taking behavior is to a large extent fairly appropriate for financial markets for standard securities, such as options, futures, stocks and bonds. The assumption of frictionless trading – however – is surely inappropriate. Margin requirements, short selling restrictions, borrowing constraints and collateral requirements belong to the basic facts of (financial) life, even for the most competitive financial markets.

In this paper we ask *whether* and *how* we can transfer the *power and the simplicity* of pricing a risky income stream by no arbitrage arguments to a world where such constraints bind investors in their portfolio decisions. It turns out that once portfolio constraints are taken into account the requirement that financial markets admit *no* arbitrage is too restrictive. We argue that the appropriate criterion we have to use in a world with frictions is a concept which we call *no unlimited arbitrage*. Constraints can lead to situations where not *all* arbitrage opportunities are eliminated in equilibrium because the constraints prevent investors to fully take advantage of them. In parallel to the frictionless world we are able to *characterize* the requirement that financial markets admit no unlimited arbitrage by the existence of positive state prices.

Since it is our aim to analyze and clarify some of the conceptual questions that arise in transferring arguments in the spirit of no-arbitrage to a framework where investors are constrained in their potential portfolio holdings, we have decided to use a framework, which has the minimal structure that is able to address the issues in a meaningful way. The reader initiated to modern asset pricing theory and security pricing might thus perhaps miss the rich stochastic structure which has become a trade mark of this literature. We present our arguments in a framework that is stripped to the bare essentials to convey the basic logic of pricing contingent claims under constraints. Our results do however not depend on the simplified framework and can easily be generalized to richer setups.

The paper is organized as follows: Since at first sight all the different contributions to the pricing problem under constraints seem to offer their own (idiosyncratic) approach we have decided to start in section 2 with a discussion of the literature to put the papers including our own contribution into perspective. Section 3 gives an exposition of the model and introduces the formal description of constraints along with some examples. Section 4 characterizes no unlimited arbitrage in terms of state prices. Section 5 concludes.

## 2. Related Research

We do not claim to be the first authors treating security pricing in the presence of portfolio constraints. In fact there is a growing literature on this topic building on a stock of seminal papers. To our best knowledge our paper is the first to propose the concept of no unlimited arbitrage as an appropriate tool for analyzing security markets with constraints[1]. Our aim is to develop a framework in which conceptual issues can be discussed in a transparent way and which is capable to bring different approaches in the literature into a unified perspective.

In the following we give an overview on the recent literature on portfolio constraints which is most closely related to the ideas discussed in our paper. We suggest classifying the papers according to two broad categories. The literature in the first category approaches the valuation problem with constraints by extending a classical paper by Harrison and Kreps (1979) (see also Kreps (1981)) which discusses the case of an unconstrained financial market. The primitives by which the problem is approached there is an abstract linear space of *net-trades* together with a linear pricing function defined on this space. These two objects reflect in an abstract way frictionless trading of arbitrary risky income streams (the linear space property) and perfect competition (the linearity of the pricing functional). It is assumed that the economy is populated by agents with preferences over net trades about which *some* general properties are known. Among these, monotonicity ("more is better") is the most important one. In this context the question is asked: When can the pricing functional together with the feasible net trades be part of an economic equilibrium, if agents are known to have these general properties (see Kreps (1981))? The answer to this question is then given by a characterization of a no arbitrage requirement via the existence of certain *state prices*. Thus the general idea is to approach the valuation problem without postulating a specific structure on agents' preferences besides of some general properties. This general idea is then extended to a world with financial constraints. The literature in the second category approaches the valuation problem by building on the analysis of solutions to an optimization problem of a representative investor, who can put his wealth into a riskless bank account and a set of risky securities the prices of which follow some stochastic process. The valuation question in this framework is answered by pricing any contingent claim using the utility gradient of the representative investor. Thus the general idea in this approach is to postulate a specific utility function and a specific stochastic model of security prices to add portfolio constraints and analyze the value of some given claim by solving the representative investor's utility maximization problem.

Our approach is in the spirit of the first category. Important papers in this literature are He and Pearson (1991), Jouini and Kallal (1995a, 1995b), and Huang (1998). Jouini and Kallal formulate their model by restricting the framework of Harrison and Kreps (1979). They take a *convex cone*[2] of net trades (instead of a linear space) and a *sublinear pricing function* (instead of a linear

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[1]However Charupat and Prisman (1997) in a critical note on a paper by Chen (1995) have pointed out conceptual problems arising by naively transferring the no arbitrage conditions from a frictionless world to a world with constraints.

[2]A non empty subset $C$ of a real vector space $V$ is called a *convex cone*, if $x \in C$, $\lambda \geq 0 \Rightarrow \lambda x \in C$, $\forall x, y \in C$ : $x + y \in C$ (see Luenberger (1969)).

one) for contingent claims as a primitive[1]. Using these primitives, no arbitrage is characterized when some general properties on investors' preferences are assumed. Our model contains the result of characterizing no arbitrage, when net trades are constrained to be in a cone as a special case. Contrary to Jouini and Kallal, we take some effort to model *in detail* the role of security prices and financial markets for the pricing of arbitrary contingent claims. We achieve this by working in a slightly less abstract framework clearly distinguishing financial markets, the prices of securities and arbitrary contingent claims which can be generated by these securities under constraints as separate objects. We are thus able to make fully transparent under which conditions linear security prices (competition) and portfolio constraints (frictions) interact to actually *imply* a sublinear pricing function for arbitrary contingent claims. Huang (1998) conducts an analysis similar to ours in an infinite horizon event tree setting for the special case of constraint sets which are cones. Our paper discusses a more general class of constraints, because there are practically important situations for which this is indeed required. Furthermore, contrary to Huang, we discuss conceptual issues at some length and analyze the relation between no arbitrage and no unlimited arbitrage. The paper of He and Pearson (1991) in contrast to ours considers a smaller class of constraints. They give a characterization of arbitrage free prices under constraints for this special case but when they make use of the characterization to value an arbitrary contingent claim, they have to use a utility function. Resorting to a utility function can be avoided in our approach. Finally a discussion most closely related to ours recently appeared in a book by LeRoy and Werner (2001)[2]. Our discussion differs from LeRoy and Werner in two aspects. We cover a larger class of constraints and our characterization of the no unlimited arbitrage condition quantifies the costs of the friction caused by portfolio constraints.

Seminal papers in the second category are by Cvitanić and Karatzas (1992, 1993). These papers consider general convex constraint sets and have inspired further research, most notably Cuoco (1997), Munk (1997, 2000), Tepla (2000) and Detemple and Murthy (1997). Cvitanić and Karatzas have developed a technique which exploits duality theory in a skillful way to get arbitrage-free prices for contingent claims in a representative investor framework with portfolio constraints, where security prices follow a Brownian motion.

In the context of the literature, our paper has the following contributions: First, we demonstrate in an elementary way how ideas of asset pricing by no arbitrage can be transferred to a world with portfolio constraints. Second, we show in a transparent way how competitive security prices and trading frictions interact to restrict the valuation of arbitrary contingent claims, thus highlighting the role played by (competitive) financial markets.

## 3. The Finance Model with Portfolio Constraints

Consider the standard general equilibrium, finance model in its simplest version. There are two dates $t = 0,1$ and a finite set $S = \{1,...,S\}$ of states of the world at date1, describing uncertainty. There is a finite set $I = \{1,...,I\}$ of investors who wish to exchange a (numèraire) good, which we could think of as income. In order to do so they can competitively trade a finite set $J = \{1,...,J\}$ of financial contracts in quantities $z$ at prices $q$ at date 0. Financial contracts are promises to some payoff of the good in the different states at date 1 and are represented by a $S \times J$ matrix $A$[3]. Investors are characterized by a continuous, strictly quasi-concave, and strictly monotone utility function $u^i : R^{S+1} \to R$ and a vector $\omega^i \in R^{S+1}_{++}$ of initial endowments of the good.

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[1]A real valued function $f$ defined on a real vector space $V$ is said to be sublinear on $V$ if $f(x+y) \le f(x) + f(y)$ for all $x, y \in V$ and $f(\lambda x) = \lambda f(x)$ for all $\lambda \ge 0$ and $x \in V$. As an example, consider any norm on $V$. By definition the norm is a real valued function, which is positive, homogeneous and fulfills the triangle inequality, hence is a sublinear function (see Luenberger (1969)).

[2]While writing this paper we were not aware of LeRoy and Werner's book. We introduced independently the same terminology to describe the consequences of portfolio constraints.

[3]We adopt the convention that all entries in A are non negative. In some parts of the finance literature such securities are called limited liability assets. Since we do not include into the model problems of bankruptcy and default, following the

For the formal description of constraints, we assume that each investor $i \in I$ can choose his *portfolios* $z$ consisting of positions in the $J$ contracts traded on the market not from $R^J$, as it is usually assumed, but only from a closed, convex set $Z \subset R^J$. To be precise we require

**Assumption (CON):** *Each Investor $i \in I$ may choose his portfolio $z^i$ from a closed, convex set $Z \subset R^J$, which is non-trivial, i.e. $Z \neq \{0\}$ and which contains $0$.*

Assumption (CON) allows describing a fairly large class of practically important restrictions on portfolio holdings. To see this, let us consider some examples.

**Example 1. *Margin Requirements*:** *Margin requirements are common practice in security trading. In particular in derivative trading investors are required to keep margin accounts, which represent a performance bond. Margins are set by regulators, clearing houses and intermediaries. Many of the common margin requirements can be described by (CON). The particular form will depend on the specific margin requirement considered. An example of a margin requirement is for instance that security positions can only be chosen from the set*

$$Z = \left\{ z \in R^J \middle| q_j z_j \geq -m_j qz \text{ for } m_j \in R_{++}, j \in J \right\}$$

*Thus the ability of investors to short sell certain securities is limited by the requirement to maintain an income margin, which is a (linear) function of their creditworthiness. Note that this example refers to margins in derivatives markets. Margins required in equity trading are a slightly different issue. There the margin has the function of a down payment for the purchase of an equity and is de facto like a loan. When we talk of margin requirements we mean margin accounts with a performance bond function as it is common for instance in futures trading.*

**Example 2. *Collateral Requirements*:** *Some securities traded on competitive financial markets can be used as debt instruments and have to be secured by an asset or a pool of assets, which are often other securities. Examples are collateralized swap contracts, collateralized mortgage obligations, collateralized depository receipts or collateralized bond obligations. One way to describe such constraints by (CON) can for instance be as follows: Let us divide a portfolio $z \in R^J$ into assets and liabilities, depending on whether $z_j > 0$ or $z_j < 0$. Denote assets by $z^+ = (\max[0, z_j])_{j=1}^J$ and liabilities by $z^- = (\min[0, z_j])_{j=1}^J$. The requirement that liabilities are partially collateralized by assets can then be written in terms of the set $Z$ as*

$$Z = \left\{ z \in R^J \middle| -qz^- \leq \theta qz^+, \theta \in [0,1] \right\}$$

**Example 3. *Portfolio Mix Constraints or Target Ratios*:** *Constraints on the mix of a portfolio or target ratios for specific assets are common in security trading. These constraints can come from various sources, for instance regulations or corporate financial policies. Whenever we have a situation where constraints of this sort occur, we can use (CON) to describe it. In this cases the set $Z$ can be described as*

$$Z = \left\{ z \in R^J \middle| \alpha q_j z_j \leq q_k z_k \leq \beta q_j z_j \text{ with } k, j \in J,\ 0 \leq \alpha \leq \beta \right\}$$

*Huang (1998) has modelled debt to equity ratios in this way. These constraints require investors to keep the ratio of asset $k$ and $j$ in a certain range determined by the bounds $\alpha$ and $\beta$.*

**Example 4. *Bid-Ask Spreads and Taxes*:** *Assumption (CON) is also able to model trading frictions expressed by different bid and ask prices, as studied for instance by Jouini and Kallal (1995b). This can be formalized by considering two financial contracts $A^j$, $A^k$, $j,k \in J$ with an identical payoff structure (i.e. $A^j = A^k$) one of which can not be sold short ($z^j \geq 0$),*

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mainstream of the literature on asset pricing, this assumption can be made without loss of generality. The results do not depend on this assumption.

*whereas there is a buying constraint on the other one ($z^k \leq 0$). As an example think of a riskless bank account, which can be used for saving and borrowing. This can be modelled as two uncontingent income streams $1 \in R^S_+$ for the savings and for the borrowing account. The savings account must not be sold short, whereas the borrowing account can only be held in negative amounts. The restriction de facto makes two different assets out of $A^j$, $A^k$ which will be reflected in different prices. The difference between these prices can be interpreted as a bid-ask spread. By the same logic one could use (CON) to describe the effects of taxes as in Prisman (1986) or in Dybvig and Ross (1986).*

From a formal viewpoint in all these examples $Z$ is a *convex cone*. This is the case almost exclusively dealt with in many papers on arbitrage and portfolio constraints. However to capture some important additional portfolio constraint situations, which are practically relevant, let us point out that our weaker requirement that $Z$ is just a closed and convex subset of $R^J$ is indeed necessary. To see this consider the following:

**Example 5. *Short Selling Limits and Buying Floors*:** *Many securities are restricted in the amount that can be sold short. Stocks can usually not be sold short in large amounts or only at a very high cost. Buying constraints can occur, when some legal restrictions prevent holding of particular securities above some given threshold prescribed by the regulation. Constraints of this nature can easily be described by (CON). Consider for example different short selling limits on securities $i = 1, \ldots, k$ and buying floors for securities $j = k+1, \ldots, J$, then*

$$Z = \left\{ z \in R^J \middle| z_j \geq l_j, z_j \leq u_i, l_i,\ u_i \in R \text{ with } j = 1,\ldots,k, i = k+1,\ldots,J \right\}.$$

For $l \neq 0$ or $u \neq 0$ the constraints do not generate a cone but rather a translation of a cone. Let $p = (l, u)$ then $Z - p$ is a cone. Following Luenberger (1969) we will call this a cone with vertex $p$. From a formal point of view these constraint sets are almost like the cones discussed in the major parts of the literature but not quite. We will see that the role played by the vertex $p$ is not as innocuous as one might assume at first sight. Note that the constraint set need not have a linear structure. If feasible portfolio holdings are functions of risk measures like Value at Risk feasible portfolio holdings might for instance become functions of volatility parameters.

**Example 6. *Capital Adequacy*:** *Constraints of the sort described in the previous example have become of particular interest during the last years, where capital adequacy has dominated the regulatory debate about financial markets. Capital adequacy is a risk management concept which requires that the capital of a financial organization is sufficient to protect its counterparties and depositors from on- and off-balance sheet market risks, credit risk, etc. The European Union has recently implemented capital adequacy rules and they have become particularly important in portfolio insurance. Capital adequacy rules work like a minimal capital requirement (Bardhan, 1994). The requirement can be a function of risk measures like for instance Value at Risk (Jorion, 1996).*

**Example 7. *Risk Based Capital Requirements*:** *Sometimes capital market regulations can lead to constraint situations, where the portfolio set $Z$ is bounded. Cvitanić (1997) gives as one particular example situations where feasible security holdings are limited in potential long and short positions. For instance the regulation of insurance companies sometimes prescribes so called risk based capital requirements. These requirements limit the amounts that can be invested into assets of a certain (default) risk class. Combined with short selling limits such constraints lead to a set $Z$, which can't be described by a cone or a cone translation.*

These two examples belong to a class of constraints which are of considerable practical importance. However in these cases $Z$ is not a cone but rather just a closed and convex set. Assumption (CON) allows describing these cases.

This discussion demonstrates that the consideration of more general constraint sets than those which are usually dealt with in large parts of the literature is indeed *required* to cover important situations occurring in the practice of financial markets.

Let us finally note that assumption (CON) is used in different, essentially equivalent, versions. For instance, a seminal paper by Cvitanić and Karatzas (1992) works with constraints on proportions of initial endowment $\omega^i$ invested in various available assets (see also the textbook by Pliska (1997)). Some authors model direct constraints on dollar amounts that can be invested. All these approaches can easily be translated into each other. In our view the description chosen here allows a particularly transparent description of the relation between competitive financial markets, portfolio constraints, state prices and the implied contingent claim values.

Note that $Z$ contains $0$. This property of the portfolio constraints is natural because a reasonable model should always allow for not making any financial market transactions and just consuming the endowment, whatever the constraints may be. As a formal object the financial market model is a tuple $\mathcal{E} = \{(u^i, \omega^i)_{i=1}^I, (A, Z)\}$.

Investors achieve consumption indirectly via competitive security trading. Because of portfolio constraints, however, each investor is confined to a restricted (future) consumption profile depending on the constraint set $Z$. If we add to $A$ as the first row the vector $-q$ to form a new matrix

$$T = \begin{bmatrix} -q \\ A \end{bmatrix} \tag{1}$$

we can write the *net income transfers* achievable for consumer $i$ by holding a portfolio $z^i \in Z$ as

$$\tau^i = Tz^i, \qquad z^i \in Z. \tag{2}$$

Using (2) we can define the feasible income transfers induced by the financial market. Let us introduce the following definition:

**Definition 1:** The set of feasible income transfers induced by $(T, Z)$ is denoted by

$$C = \left\{ \tau \in R^{S+1} \middle| \tau = Tz, \quad z \in Z \right\}.$$

Since $Z$ is a closed, convex set, and $T$ is a continuous linear transformation, the set of achievable income transfers will also be convex. Indeed we can assert:

**Lemma 1:** The set $C$ of feasible income transfers is a convex subset of $R^{S+1}$ containing $0$.

**Proof:** $C = \left\{ \tau \in R^{S+1} \middle| \tau = Tz, \quad z \in Z \right\} \subset R^{S+1}$ by Definition 1. Since $0 \in Z$ and $T$ is a linear transformation $C$ must also contain $0$, hence be non-empty. Let $\tau_1$ and $\tau_2$ be in $C$ and consider $\lambda \in [0,1]$. Then $\lambda\tau_1 + (1-\lambda)\tau_2 = \lambda T z_1 + (1-\lambda)Tz_2 = T(\lambda z_1 + (1-\lambda)z_2)$. Since $Z$ is convex, $\lambda z_1 + (1-\lambda)z_2$ is in $Z$, thus $\lambda\tau_1 + (1-\lambda)\tau_2$ is in $C$ and $C$ is convex.

Unfortunately the set $C$ does not necessarily inherit the closedness of $Z$ [1]. We preclude such a situation by assumption.

**Assumption (CONC):** *Each Investor* $i \in I$ *may choose his portfolio* $z^i$ *from a closed, convex set* $Z \subset R^J$ *, which is non-trivial, i.e.* $Z \neq \{0\}$ *and which contains* $0$. $Z$*,* $A$ *and* $q$ *are such that* $C$ *is closed* [2].

Since many practically important portfolio constraints can be formally described as cones or translations of cones, we want to know whether the set of feasible income transfers inherits this structure from $Z$.

[1] Sufficient conditions that *C* will inherit the closedness property from *Z* under the mapping *T* would for instance include the cases where *Z* is compact or polyhedral.

[2] An assumption similar to this is used in a different context by Ross (1987).

**Lemma 2:** *If $Z$ is a convex cone with vertex $p$, $C$ is a convex cone with vertex $Tp$.*

**Proof:** If $Z$ is a cone, $z \in Z$ implies that for all $\lambda \geq 0$ the vector $\lambda z \in Z$. Thus $Tz \in C$ implies $T(\lambda z) = \lambda Tz \in C$, thus $C$ is a cone. If $Z$ is a cone with vertex $p$, then $Z - p$ is a cone. Hence the set $C' = \{\tau \in \mathbb{R}^{S+1} \mid \tau = Tx, x \in Z - p\}$ is a cone. As $C = C' + Tp$ it is a cone with vertex $Tp$.

## 4. Arbitrage and Portfolio Constraints

A central idea of modern asset pricing theory is the explanation of the value of securities by analyzing security prices $q \in R^J$ which allows *no arbitrage*. Essentially this requirement expresses the idea that in any *equilibrium*[1] it should not be possible to achieve a consumption bundle at zero costs that increases some investor's utility by trading securities (Kreps, 1981). The reason is that investors with monotone preferences would then wish to take an unlimited position in the arbitrage portfolio to generate an unlimited consumption profile. The ability to take arbitrary portfolio positions is therefore one essential building block of pricing arguments which invoke a no arbitrage condition.

Though ultimately the pricing of risky income streams by no arbitrage indirectly relies on an equilibrium argument, much of its power comes from the fact that the question whether or not security prices are arbitrage free (and can therefore be part of some equilibrium) can be inferred from the payoff structure of securities, the matrix $A$, and the observed security prices $q$. There is no need to know the entire equilibrium. A famous theorem of finance (see for instance Duffie (1996)) demonstrates for the standard finance model that the absence of arbitrage is equivalent to the existence of implicit strictly positive values of income in the different states of the world – the so called *state prices* – such that the value of any security is exactly equal to the value of the future income stream it provides under these state prices. It can therefore be checked from $(q, A)$ alone whether or not there is an arbitrage possibility. On this limited information it is thus possible to find out which $q's$ are consistent with some equilibrium. Strictly positive state prices which make securities zero-profit investments, *characterize* the absence of arbitrage in the standard finance model.

Most of the models transferring this kind of argument to a world with constraints work with a generalization exactly along these lines. It can be formulated in analogy to the unconstrained case: The financial market $(q, A, Z)$ allows no-arbitrage if there exists no $z \in Z$ such that $Tz > 0$[2]. Written in a slightly less condensed form the definition requires that there does not exist a $z \in Z$ with $-qz \geq 0$ and $Az \geq 0$ where at least either the first or one of the other $S$ inequalities is strict.

However, when we consider general situations of convex constraints as formalized by assumption (CON) and illustrated by the various examples we gave before we have to be careful since such a characterization might be too strong and we need a slightly *weaker* criterion. The problem which arises when we consider constraints in the class described by (CON) can perhaps most clearly be seen in a toy example where $Z$ is a cone with vertex $p$. The basic message of the example is that portfolio constraints can lead to a situation where security prices can allow in principle financial transfers that imply *limited* arbitrage opportunities. This situation can nevertheless be consistent with *some* equilibrium, since constraints make it impossible for individuals to take any advantage of them.

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[1]For a formal definition of an *equilibrium* for a model with financial markets with no portfolio constraints see for instance Magill and Quinzii (1996).

[2]The inequality $x > 0$ for a vector $x \in R^n$ means that all components of the vector are nonnegative and not all of them are zero. We have $x > 0 \Leftrightarrow x \in R^n_+$ and $x \neq 0$.

**Example 8:** *Consider a model with no uncertainty, so that* $S=\{1\}$. *There are two investors* $i=1,2$ *with endowments* $\omega^1=(8,1)$ *and* $\omega^2=(2,14)$ *who have both identical preferences described by the utility function*

$$u^i(x_0^i,x_1^i)=\log(x_0^i)+\log(x_1^i)$$

*The payoff matrix of financial contracts is given by* $A=(1,1)$ *and the constraint set is given by* $Z=[-2,\infty)\times(-\infty,2]$. *So there is a short selling limit on security one and a buying floor on security two. Now it is easy to check that the security prices* $q^*=(1,1/2)$ *and the consumption and security demands* $x^{1*}=(5,5)$ *and* $x^{2*}=(5,10)$, $z^{1*}=(2,2)$ *and* $z^{2*}=(-2,-2)$ *form an equilibrium for this economy because at these prices each investor has solved his utility maximization problem and in the market for the good and for securities supply and demand are balanced*[1]. *The example can perhaps most clearly be seen by looking at Figure 1, which shows the equilibrium.*

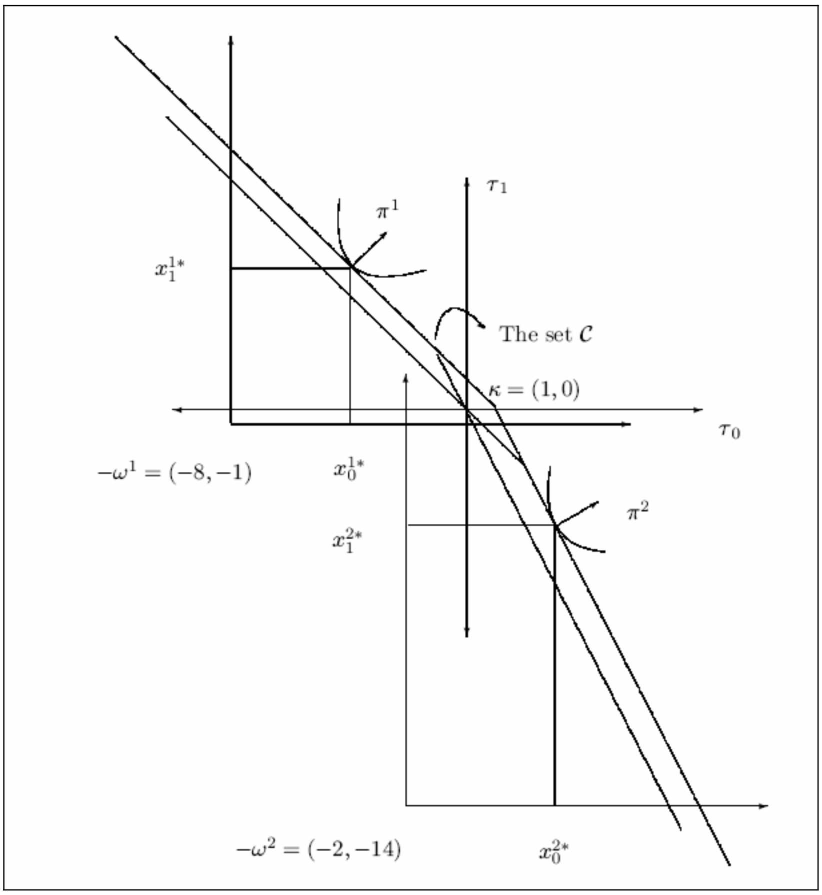


Fig. 5. With portfolio constraints the financial markets need not be arbitrage free even in equilibrium

[1]To calculate an equilibrium apply Definition 8.2. p. 69, in Magill and Quinzii (1996), replacing the condition $z^i \in R^J$ by the condition $z^i \in Z$.

What can we learn from this example? Had we required that there exists *no* $z \in Z$ with $Tz > 0$ to exclude all prices which can't possibly be part of any equilibrium, we would have obviously discarded the prices $q^* = (1, 1/2)$. However these prices – as we have just seen – are consistent with some equilibrium and should therefore not be ruled out. Exactly this would have occurred, however, by applying the criterion of *no arbitrage*. From the example we can see that it is *not* necessary for an equilibrium that there exists no $z \in Z$ such that $Tz > 0$. Obviously such $z's$ do exist and yet we have an equilibrium because constraints prevent the advantageous use of these opportunities by the agents of the economy[1].

One can take a geometric viewpoint on the requirement of no arbitrage. It is equivalent to the condition that $C \cap R_+^{S+1} \setminus \{0\} = \varnothing$. This condition is not fulfilled in the equilibrium of the example because there $C$ is a cone with vertex $\kappa = (1,0)$. Therefore $C$ can not have an empty intersection with $R_+^{S+1} \setminus \{0\}$. This example suggests the following *weaker* criterion for a world with portfolio constraints described by (CONC).

**Definition 2 (NUA):** The financial market $(q, A, Z)$ allows no unlimited arbitrage if there exists a vector $\kappa \in C$ such that there is no $z \in Z$ with $Tz > \kappa$.

In analogy to the unconstrained world we can *characterize* this requirement by the existence of certain state prices. We assert the following

**Theorem 1:** $(q, A, Z)$ admits no unlimited arbitrage if and only if there exist a vector $\overline{\pi} \in R_{++}^{S+1}$ and a vector $\tau^* \in C$ such that $\overline{\pi}\tau \leq \overline{\pi}\tau^*, \forall \tau \in C$.

**Proof:** Let $\mathrm{T} = \{\tau^* \in C \mid \text{there is no } \tau \in C \text{ such that } \tau > \tau^*\}$. Assume $(q, A, Z)$ admits no unlimited arbitrage. Then T is not empty. Denote by $\mathrm{T}int(\mathrm{T})$ the interior of $\mathrm{T}$ with respect to the topology induced on T, take some $\tau^* \in \mathrm{T}int(\mathrm{T})$ and consider the set $C' = \{\tau \in R^{S+1} \mid \tau + \tau^* \in C\}$. Since we have assumed (NUA), $C' \cap R_+^{S+1} \setminus \{0\} = \varnothing$. $C'$ is a non-empty, closed, convex, set in $R^{S+1}$ since it is a translation of the set $C$, which is non-empty, closed and convex by Lemma (1) and has a non-empty interior by (CONC). Let $\Delta$ be the non-negative simplex in $R^{S+1}$. The simplex is a convex and compact subset of $R^{S+1}$ containing no interior points of $C'$ since we have assumed that there is no unlimited arbitrage. We can therefore apply a version of the separating hyperplane theorem (Magill and Quinzii, 1996, p. 73). The separation theorem implies that there is a linear functional $0 \neq \overline{\pi} \in R^{S+1}$ such that

$$\sup_{\tau \in C'} \overline{\pi}\tau < \inf_{\tau \in \Delta} \overline{\pi}\tau$$

Let $K(C') = \{\lambda\tau \mid \tau \in C' \text{ and } \lambda \in R^{S+1}\}$ the convex cone generated by $C'$. This cone is non-empty, closed and convex. As $0 \in \mathrm{T}int(\mathrm{T}) - \tau^*$ by assumption, $K(C') \cap R_+^{S+1} = \{0\}$. Therefore the above separating hyperplane theorem is applicable. As $C' \subset K(C')$, the set of $\pi's$ that separate $K(C')$ from $\Delta$ also separate $C'$ from $\Delta$. Hence it

[1]We have to mention that problems of equilibrium mispricing with features similar to those in our example have been already pointed out in the early literature on tax arbitrage (Damon and Green (1987), Ross (1987)). In the context of portfolio constraints this problem has been described by Basak and Croitoru (2000) and in an earlier paper by Charupat and Prisman (1997). These results – in particular the paper by Charupat and Prisman (1997) – seem to have remained largely unnoticed in the literature on asset pricing with portfolio constraints.

suffices to show that there is a $\bar{\pi} \in R_{++}^{S+1}$ separating $K(C')$ from $\Delta$ such that $\bar{\pi}\tau \leq 0$ for all $\tau \in K(C')$.

Since $0 \in K(C')$, the separation inequality implies that $0 < \inf_{\tau\in\Delta} \bar{\pi}\tau$. Suppose now that $\bar{\pi}_s \leq 0$ for some state $s \in S$ and consider the transfer $e_s = (0,...,0,1,0,...,0) \in \Delta$ which is $1$ in state $s$ and $0$ otherwise. Then $0 < \inf_{\tau\in\Delta} \bar{\pi}\tau \leq \bar{\pi}e_s = \bar{\pi}_s \leq 0$ which is a contradiction. Hence $\bar{\pi} \in R_{++}^{S+1}$. It remains to be shown that $\bar{\pi}\tau \leq 0, \ \forall \tau \in K(C')$. Suppose there is a $\bar{\tau} \in K(C')$ such that $\bar{\pi}\bar{\tau} > 0$. Since $K(C')$ is a cone it follows that $\lambda\bar{\tau} \in K(C')$, $\forall \lambda \geq 0$. It is easy to see that $\inf_{\tau\in\Delta} \bar{\pi}\tau = \min(\bar{\pi}_1,...,\bar{\pi}_{S+1})$, which is finite. Now we can choose a sufficiently large $\lambda$ such that $\lambda\bar{\pi}\bar{\tau} > \inf_{\tau\in\Delta} \bar{\pi}\tau$. This is a contradiction.

Contrary to the result for unrestricted economies we can not turn the inequality into an equality. The obvious reason is that $\tau \in C'$ does *not* imply $-\tau \in C'$. Thus we get only the inequality $\bar{\pi}\tau \leq 0, \ \forall \tau \in C'$ which can be rewritten as $\bar{\pi}\tau \leq \bar{\pi}\tau^*, \ \forall \tau \in C$.

To prove the other direction assume there is $\tau^* \in C$ and $\bar{\pi} \in R_{++}^{S+1}$ such that $\bar{\pi}\tau \leq \bar{\pi}\tau^*, \ \forall \tau \in C$. This implies $\bar{\pi}(\tau - \tau^*) \leq 0 \ \ \forall \tau \in C$. Because $\bar{\pi} \in R_{++}^{S+1}$ we can conclude that $\tau - \tau^* > 0$ is impossible for all $\tau \in C$. And this is exactly the no unlimited arbitrage condition.

**Remark:** We have imposed assumption (CONC) to cover a large class of practically important constraint situations. Unfortunately some rather strange cases are also compatible with (CONC) and therefore we have to use the $\mathrm{T}int(\mathrm{T})$ construction. Why do we have to take $\tau^* \in \mathrm{T}int(\mathrm{T})$? If we took $\tau^* \in \mathrm{T}$ but not in $\mathrm{T}int(\mathrm{T})$, we could face the problem that $K(C')$ meets $R_{+}^{S+1}$ not only in $0$ but also in other points $x \geq 0$ but not strictly larger than $0$. Hence we would get the weaker result that there has to exist a $\bar{\pi} \in R_{+}^{S+1}$ $\bar{\pi}$ such that $\bar{\pi}\tau \leq \bar{\pi}\tau^*, \ \forall \tau \in C$, where some components of $\bar{\pi}$ might be $0$.

What does this theorem tell us about security prices that can be part of some equilibrium? To get an economic interpretation it is useful to write the inequality, which appears in the characterization of our theorem in a slightly less condensed form. In order to do so, let us normalize $\bar{\pi} = (\bar{\pi}_0, \bar{\pi}_1, \ldots, \bar{\pi}_S)$ by the first entry $\bar{\pi}_0$ so that we get

$$\left(\frac{1}{\bar{\pi}_0}\right)\bar{\pi} = \left(1, \frac{\bar{\pi}_1}{\bar{\pi}_0}, \ldots, 1, \frac{\bar{\pi}_S}{\bar{\pi}_0}\right) = (1, \hat{\pi}_1).$$

Let $\tau^* = (\tau_0^*, \tau_1^*)$. Then the no unlimited arbitrage inequality can be written as

$$qz \geq \hat{\pi}_1 Az - (\tau_0^* + \hat{\pi}_1\tau_1^*) \quad \forall z \in Z, \tau^* \in C. \tag{3}$$

In general in a situation with convex constraints of the portfolio holdings the present value of any traded security must be *larger or equal to the present value of the income stream under the state prices* $\hat{\pi} = (1, \hat{\pi}_1)$ *corrected by the present value of additional transfers provided by the limited arbitrage possibility induced by an appropriately chosen* $\tau^* \in C$.

As the limited arbitrage opportunity can be exploited only once the pricing functional need not be sublinear. Only in the case where $\tau^*$ equals **0** we get the result that the pricing functional has to be sublinear.

A problem with this characterization is obviously that typically there are many $\tau^*$ and $\overline{\pi} \in R_{++}^{S+1}$ fulfilling the inequality $\overline{\pi}\tau \leq \overline{\pi}\tau^*, \forall \tau \in C$. Moreover these $\overline{\pi} \in R_{++}^{S+1}$ will differ depending on the chosen $\tau^*$. Let $\Pi(\tau^*)$ denote the set of all $\overline{\pi} \in R_{++}^{S+1}$, which fulfill the inequality $\overline{\pi}\tau \leq \overline{\pi}\tau^*, \forall \tau \in C$ for a given feasible $\tau^* \in C$. The set of *all* potential state prices $\Pi$, which are consistent with the requirement of no unlimited arbitrage, is characterized by the union $\bigcup_{\tau^* \in C} \Pi(\tau^*)$, i.e. $\Pi = \bigcup_{\tau^* \in C} \Pi(\tau^*)$. If worse comes to worst, this union might be rather too large to provide a reasonably sharp characterization. Moreover the whole construction looks quite bulky at first sight. These remarks can perhaps best be seen in Figure 2, illustrating an example with so called "rectangular constraints" (Cvitanić 1997).

The examples have shown that many of the practically important constraint situations described by (CONC) are cones or translations of them. For this important class of constraints we are able to give a characterization of no unlimited arbitrage by using the vertex $Tp$ of $C$.

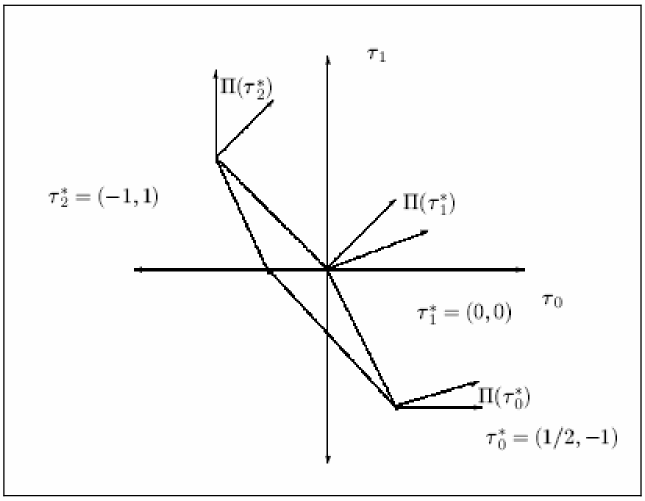


Fig. 6. Rectangular Constraints

**Corollary 1:** *If $Z$ is a cone with vertex $p$, the financial market $(q, A, Z)$ admits no unlimited arbitrage if and only if there exists a vector $\overline{\pi} \in R_{++}^{S+1}$ such that $\overline{\pi}\tau \leq \overline{\pi}Tp, \forall \tau \in C$.*

**Proof:** It is sufficient to show that $\Pi(\tau) \subset \Pi(Tp) \quad \forall \tau \in C$. If $\Pi(\tau^*) = \varnothing$ our assertion is trivially true. Now take any $\tau^*$ such that $\Pi(\tau^*) \neq \varnothing$ and suppose that there exists a $\overline{\pi} \in \Pi(\tau^*)$ such that $\overline{\pi} \notin \Pi(Tp)$ and hence $\overline{\pi}Tp < \overline{\pi}\tau^*$. Let $\tau^+ = (1+\varepsilon)(\tau^* - Tp) + Tp = \tau^* + \varepsilon(\tau^* - Tp)$ where $\varepsilon > 0$. By construction $\tau^+ \in C$. Premultiplying $\tau^+$ by $\overline{\pi}$ yields

$$\overline{\pi}\tau^+ = \overline{\pi}\tau^* + \varepsilon(\overline{\pi}\tau^* - \overline{\pi}Tp) > \overline{\pi}\tau^*,$$

where the inequality follows from the assumption that $\overline{\pi} \notin \Pi(Tp)$ . This result contradicts that $\overline{\pi} \in \Pi(\tau^*)$ . Hence $\Pi\left(\tau^*\right) \subset \Pi(Tp)$ .

**Remark:** *As* $Tp$ *is not necessarily in* $Tint(T)$ *this Corollary implies that* $\Pi(Tp) \neq \varnothing$ *if there is a* $\tau^*$ *such that* $\Pi(\tau^*) \neq \varnothing$ *, i.e. if the market admits no unlimited arbitrage.*

Thus for constraint sets which are cones with vertex $p$ we know that all state prices which allow no unlimited arbitrage must lie somewhere in the set.

$$(C - Tp)^! := \left\{\pi \in R^{S+1} \mid \pi\tau \leq \pi Tp, \forall\, \tau \in C\right\}. \tag{4}$$

which is just the negative conjugate cone of $C - Tp$ [1]. Thus in the case of cones or cone translations the set of all candidate state prices can be constructed by polarity from the knowledge of the set $C$ of feasible income transfers induced by financial markets.

This important special case, gives us an opportunity to relate our results to other characterizations of no arbitrage in the presence of constraints, which have been suggested in the literature.

The cases almost exclusively dealt with are constraint sets, which are cones. In particular this case is investigated in the papers by He and Pearson (1991), Jouini and Kallal (1995a), Jouini and Kallal (1995b) and Huang (1998).

For cones the condition of no unlimited arbitrage reduces to the familiar requirement of *no* arbitrage, which is stated in the following

**Corollary 2:** *If* $Z$ *is a cone,* $(q, A, Z)$ *admits no unlimited arbitrage if and only if there exists a vector* $\overline{\pi} \in R^{S+1}_{++}$ *such that* $\overline{\pi}\tau \leq 0, \forall\, \tau \in C$ .

**Proof:** This is a direct consequence of Theorem 1 and Corollary 1.

By (4) we see that in this case we get a direct generalization of the results for a world without frictions. There the set of income transfers is a linear space and its polar cone is its orthogonal complement.

If we normalize $\overline{\pi}$ as above, the *no-arbitrage* inequality implies that security prices must fulfill the relation:

$$qz \geq \hat{\pi}_1 Az \quad \forall z \in Z \tag{5}$$

for a strictly positive $\hat{\pi}_1$ [2].

The economic interpretation of this inequality is that security prices are arbitrage free if and only if they are *larger or equal to the present value of the future income stream* provided by the corresponding securities traded on the financial markets. Potential discrepancies do not generate arbitrage opportunities because the constraints on the feasible portfolio positions prevent investors from taking advantage of these opportunities.

This is the result which has repeatedly been obtained in the literature on portfolio constraints independently by He and Pearson (1991) and by Jouini and Kallal (1995a) and Huang

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[1] Let $X$ be a vector space which is equipped with the inner product $\left\langle x, x^*\right\rangle$ . Denote by $X^*$ its dual space. Given a set $S \subset X$ , the set $S^! = \left\{x^* \in X^* : \left\langle x, x^*\right\rangle \leq 0 \text{ for all } x \in S\right\}$ is called the negative conjugate cone of $S$ (Luenberger, 1969).

[2] If some asset $j$ must not be sold short (i.e. $z_j \geq 0$ ) the price of the asset must not be less than the promised income stream evaluated at the state prices $\overline{\pi}_1$ (i.e. $q_j \geq \sum_{i=1}^{S} \overline{\pi}_{1,i} a_{j,i}$ ). This is exactly the characterization given in LeRoy and Werner (2001, Theorem 7.3.2).

(1998). Our discussion shows that in fact a constraint set which is a cone is needed to get (5) as an *appropriate* characterization of prices which can be part of some equilibrium. It can be seen as a special case of no unlimited arbitrage with a vertex $\kappa = \mathbf{0}$.

Our inequality for cones is exactly the statement of Jouini and Kallal's Theorem 2.1 (Jouini and Kallal, 1995a). Their theorem says that no arbitrage is equivalent to the existence of a positive linear functional (in our case $\pi$), with the property that its restriction to $C$, lies below the contingent claim pricing functional. This is exactly what is expressed in condition (5). Though we don't know the pricing functional for an arbitrary income stream $y \in C$ yet, we know that in any way it must fulfill inequality (5). We see that more generally this condition has to be modified and can be appropriately applied only for the case of constraints which are a cone.

## 5. Conclusions

In this paper we have analyzed the pricing of contingent claims in security markets, where investors are constrained by various trading restrictions, which can be described as a convex set. We have given a characterization of no unlimited arbitrage in the *simplest* possible framework. As it turns out a financial market allows no unlimited arbitrage if and only if there exists present value of any traded security must be larger or equal to the present value of the income stream under the state prices corrected by the present value of additional transfers provided by the limited arbitrage possibility induced by an appropriately chosen.

We hope that these results will prove useful for financial economists who are interested to get an overview of the economics of portfolio constraints and security pricing, without going into the technicalities of continuous time stochastic finance. Ultimately, however, this is a paper about asset pricing. It is our hope that our results will prove useful for experts interested in practically developing and implementing valuation techniques inspired by tools from a heavenly linear space, within the constraints of an earthly convex set.

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