OPTIONS ON PENSION ANNUITY

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Abstract

We introduce a European (exotic) call option on a pension annuity. The option gives its owner the right to buy, for a specified lump sum, an ordinary annuity (pension) that starts at a specified future date ("retirement age"). Thus, instead of contributing monthly to a pension fund, one could buy this option and insure the terms of their retirement.

We price the options under stochastic interest rates in both discrete and continuous time regimes. In the discrete time case, we use an order 2 autoregressive process (*AR(2)*). In the continuous time case, we use simulations of the short interest rate according to various models, such as CIR, and simulations of *GARCH* process estimated from real data.

Key words: Autoregressive Process *AR(2)*; *GARCH* Autoregressive Process; European call option; Stochastic Interest Rates.

JEL Classification: G22.

1. Introduction

We propose (exotic) call options on pension annuities, which are, in fact, pension insurance. We price them under stochastic interest rates discrete and continuous time regimes, using real market data. Options on pension annuities provide pension insurance, and have recently been discussed by Milevsky and Promislow (Milevsky and Promislow, 2001) and Yosef, Benzion, and Gross (Yosef, Benzion, and Gross, 2003). Others, including Lee (Lee, 2001), Cairns (Cairns, 2002), Ballotta and Haberman (Ballotta, and Haberman, 2003), Pelsser (Pelsser, 2003), Wilkie, Waters, and Yang (Wilkie, Waters, and Yang, 2003), and Boyle and Hardy (Boyle and Hardy, 2003) investigated similar options, called guaranteed annuity options (GAO's).

The GAO option gives the option holder the right to receive at retirement the greater of a cash payment equal to the current value of the investment in the equity fund and the expected present value of the life annuity obtained by converting this investment at the guaranteed rate. The GAO option differs from the option that we propose in that it further gives the policyholder the right to choose either the annuity based on current market rates, or an annual payment using the guaranteed annuity interest rate.

A call option on a pension annuity replaces in effect the traditional pension insurance model. Under the traditional model a person pays monthly premia into his pension fund throughout his working life, and receives a monthly benefit of B starting at a predefined age of retirement A_r. Under the plan that we propose, a person aged x, $x \leq A_r$, buys a European call option on his pension annuity. This call option allows the option holder to buy his pension annuity at a specified strike price, at or prior to the age of retirement A_r. The option holder is entitled to receive the pension benefits only if the insured is still alive at retirement age, and then throughout the insured's life in retirement. Thus instead of contributing monthly payments to their pension fund throughout their working lives, individuals could buy our proposed option as insurance, replacing the monthly premia by a two-installment system: one at age x when the option is bought, and one at exercise time at or prior to retirement age, when the option is exercised at the prespecified strike price.

Yosef, Benzion, and Gross, (Yosef, Benzion, and Gross, 2003) used a constant interest rate to simplify the pricing of options on traditional pension contracts in discrete time. In this pa-

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per we propose to extend their work in two directions: we replace the constant interest rate by stochastic interest rates and we do this, both in discrete and continuous time regimes. Milevsky and Promislow, (Milevsky and Promislow, 2003) considered both stochastic discrete and the continuous interest rates, but treated primarily options on pure endowments.

Existing treatments, including those of GAO's, evaluate pension annuity options using a theoretical Martingale measure that is hard to estimate from observed data. Their formulae depend directly on the special properties of the Martingale measure and do not hold for the empirical measure associated with real world interest rates. We provide a valuation of annuity options that is computable from the empirically estimable short rate process and its associated term structure of interest rates. Our formulae are general and do not depend on any particular model used to fit empirically observed interest rates or the term structures of interest rates. They provide a model-independent computational vehicle rather than an explicit formula. The convergence of infinite sums involved in the discrete time case, and the integrals in the continuous time case, depends only on the finite lifetime of the insured, and requires no extra assumptions.

We first present discrete (Section 2) and continuous time (Section 3) formulae for computing the value of a traditional pension insurance in which the insured pays a single premium at age x to receive a monthly pension benefit B from the age of retirement A_r through death, under any stochastic structure for the short and term structure of interest rates. We then use the formulae to price options on pension annuity plans.

We apply these formulae to simulated data from discrete and continuous time models. The data on which we base our simulations is the monthly Bank of Israel nominal bank rate series from 1/1985 to 12/2002 for the discrete time simulations. For the continuous time we use the monthly Eurodollar annualized deposit interest rates for 4/1953 through 5/2003 published by the Federal Reserve in its H.15 database. The choice of data is quite arbitrary, except that it serves to fit a stochastic model for the short rates from two different sources, providing estimated parameter values for the simulations that follow. We then present tables for the pension annuity option mean values and standard deviations as a function of insured age, strike price, strike time, mean short interest rate, and interest rate process volatility.

A constant volatility discrete autoregressive process [See (Panjer and Bellhouse, 1980) and (Parker, 1994)] of order two adequately fits the Israeli data. For the Eurodollar data we tried various term structure models based on stochastic volatility short rate models found elsewhere in the literature [e.g., CIR model,)Cox, Ingersoll, and Ross, 1985)], but found that none of these models fit our empirical data very well. Chan, Karolyi, Longstaff and Sanders (Chan, Karolyi, Longstaff and Sanders, 1992) have obtained similar results for different empirical data. We therefore resorted to the stochastic volatility *ARCH-GARCH* discrete approximations to the continuous models. The parameters we estimated for the *ARCH-GARCH* models were then used for generating data in the continuous time simulation.

We assumed that real world investors have observed the short rate structure for a long time and have learnt the laws of motion and the parameters of the dynamics of the short rate. These laws induce expectations about future short rates that in turn induce a term structure with implied forward rates, which we use in pricing or discounting cash flows. In other words, we use empirical data to estimate the real-world short rate process, and use these expectations to determine the term structure and its associated forward rates. Because the Eurodollar deposit rate (or Bank of Israel interest rate) process embodies the same risk levels as the cash flows we priced, we can now use the implied forward rates $r_{-2}, r_{-1}, \ldots$ to price our cash flows.

We emphasize that premia and annuity option prices depend on two stochastic phenomena: the interest rate process and the insured lifetime, which itself may be modeled as a stochastic process because lifetime distributions may vary in time. In our simulations we simulated stochastic rate processes using estimated parameters from empirical data. However, rather than model the survival distribution, we used published Israeli and US life-tables for computing conditional survival probabilities in the time-independent, non-dynamic survival case. Life tables used by the Israeli insurance industry until approximately ten years ago were in fact British tables. The most commonly used table has been the A (1967-1970) table, which we use in our simulations in the discrete time case. In the continuous time simulations we used both American life tables, and American interest rate data. American life data were obtained from the US Decennial life tables for 1989-1991, table for the total population, for computing conditional survival probabilities in the time-independent, stochastic survival case.

The choice of life-tables over specific probability distributions, or specific dynamic survival processes, as was done by Milevsky and Promislow (Milevsky and Promislow, 2001), was indicated by the fact that insurance companies traditionally use life-tables rather than fitted survival distributions. Also, the presentation of our formulae in terms of life-table, time-invariant, and conditional probabilities, permits a simple computational adjustment to incorporate time dependent life-table conditional probabilities. We shall indicate the changes required in the presentation that follows.

Finally, we note that other contexts in which two or more stochastic phenomena govern the life of a call option have been studied in the literature. Two important ones are options on defaultable bonds [see e.g., Duffy and Singleton (Duffy and Singleton, 1997)], and options on Asian exchange rates in a two-currency economy [see e.g., (Nielsen and Sandmann, 2002)]. In the first, the default hazard is used to adjust the instantaneous interest rate. In the second, both the foreign and domestic currency zero-coupon bond prices are used to model the exchange rates. In both cases the *two stochastic phenomena are assumed to be independent*, even though the assumption is questionable in both cases. It is far more reasonable in our case.

Our presentation is divided into discrete and continuous time modeling of forward rates, with corresponding formulae and simulations. The models actually employed in the discrete case are fixed volatility autoregressive processes, whereas the discrete approximations used in the continuous time forward rate processes are stochastic volatility models. Nonetheless, a quick comparison of results presented in Tables 2 and 7 for the two cases, for identical ages at annuity option purchase, identical strike prices and average forward rates, are rather similar. This is true despite the fact that the sources of empirical data from which volatility parameters were estimated are different, and the first is based on a constant volatility autoregressive process of order two, and the second is based on a stochastic volatility, autoregressive model of order two. The life tables used in the two cases belong of course to the different countries of origin of the data.

Our simulation results display mean and standard deviations for annuity option prices as functions of age at annuity option purchase, strike price, and average forward rate. We emphasize that the standard deviations reported are the standard deviations of the (stochastic) option price as estimated from the simulation. We found it to be very stable as the number of simulations increased. It is an inherent property of the option price that depends on the two stochastic phenomena of age at death and forward interest rates. This is the case because we used parameter values that were estimated from real data in two very different countries and economies. No amount of extra simulations will reduce these standard deviations. These relatively large standard deviations are an inherent property of options on annuities in a stochastic interest rate economy.

2. Pricing Traditional Pension Annuity Plans: The Stochastic Discrete Force of Interest Case

Consider a traditional pension insurance in which an insured person, age x, buys pension insurance with a single premium, guaranteeing a B annuity from the age of retirement, A_r, through death. Assume that interest rates change at discrete time points, and let the series $\ldots r_{-2}, r_{-1}, r_0, r_1, r_2, \ldots, r_t$ denote the stochastic series of forward interest rates, where $t = 0$ at age x of the insured. The risk embodied in these rates is the risk of cash flows we price. It is convenient at this point, foreseeing the need to model the stochastic forward interest rate, to introduce instead the force of interest series

$$\delta_t = \log(1 + r_t), \tag{1}$$

and the cumulative force of interest

$$\Delta_t = \sum_{i=1}^{t} \delta_i, \tag{2}$$

which are not necessarily positive quantities bounded between zero and one, and thus easier to model. See Panjer and Bellhouse (Panjer and Bellhouse, 1980) and Parker (Parker, 1994).

2.1. Pricing Pension Plans Under Discrete Stochastic Interest Rates

Given the series of cumulative force of interest, the single premium paid by an insured aged x, to retire at age A_r, A_r > x, and receive B monthly thereafter until the age at death T, denoted by ${}_{A_r-x}S_x$, is given by

$${}_{A_r-x}S_x = B \sum_{A_{r \le t \le T}} \exp[-\Delta_{t-x}], \tag{3}$$

where $\Delta_0, \Delta_1, \Delta_2, \ldots, \Delta_{-1}, \Delta_{-2}, \ldots$ denote the stochastic series of forces of forward interest rates, which we shall now denote simply by Δ. Notice that ${}_{A_r-x}S_x$ is random and its expected value is, by the chain rule of conditional expectations, and the independence assumed between Δ and T,

$$E[{}_{A_r-x}S_x] = E_\Delta[E_T({}_{A_r-x}S_x \mid \Delta)], \tag{4}$$

where E_Δ refers to the expectation with respect to the stochastic force of interest, and E_T refers to the expectation with respect to the age at death whose conditional probability, given the person had already survived till age x, that he will survive beyond A_r is denoted, as usual, by ${}_{A_r-x}p_x$. We emphasize that ${}_{A_r-x}S_x$ is a complicated (not multiplicatively separable) function of T and of Δ. Therefore, even though we assume that T and Δ are independent, we cannot use the product rule for expectations, and we resort instead to the use of the chain rule. We then obtain

$$E[{}_{A_r-x}S_x] = E_\Delta\left[B\,{}_{A_r-x}p_x \sum_{t=0}^{\infty}\left({}_{(A_r+t)}p_{A_r}\right)e^{-\Delta_{A_r-x+t}}\right] = B_{(A_r-x)}p_x \sum_{t=0}^{\infty}\left({}_t p_{A_r}\right)M_{\Delta_{A_r-x+t}}(-1), \tag{5}$$

where $M_{\Delta_t}(u)$ denotes the moment generating function of Δ_t at u. Furthermore,

$$VAR\left[{}_{A_r-x}S_x\right] = VAR_\Delta\left[E_T\left({}_{A_r-x}S_x\right) \mid \Delta\right] + E_\Delta\left[VAR_T\left({}_{A_r-x}S_x \mid \Delta\right)\right], \tag{6}$$

see Ross (2001). From (3) we compute

$$E_\Delta\left[E_T\left({}_{A_r-x}S_x \mid \Delta\right)^2\right] = B^2\left({}_{A_r-x}p_x\right)^2 \sum_{t=0}^{\infty}\sum_{t'=0}^{\infty}\left({}_t p_{A_r}\right)\left({}_{t'} p_{A_r}\right)M_{\Delta_{A_r-x+t}+\Delta_{A_r-x+t'}}(-1), \tag{7}$$

$$E_\Delta\left[E_T\left({}_{A_r-x}S_x \mid \Delta\right)^2\right] = B^2\left({}_{A_r-x}p_x\right)^2 \sum_{t=0}^{\infty}\sum_{t'=0}^{\infty}\left({}_t p_{A_r}\right)\left({}_{t'} p_{A_r}\right)M_{\Delta_{A_r-x+t}}(-1)M_{\Delta_{A_r-x+t'}}(-1), \tag{8}$$

and

$$VAR\left[{}_{A_r-x}S_x \mid \Delta\right] = E_T\left({}_{A_r-x}S_x^2 \mid \Delta\right) - \left(E_T\left({}_{A_r-x}S_x \mid \Delta\right)\right)^2$$

$$= B^2({}_{A_r-x}p_x)^2\left\{\sum_{t=0}^{\infty}\left({}_t p_{A_r}\right)e^{-2\Delta_{A_r-x+t}} - \left(\sum_{t=0}^{\infty}\left({}_t p_{A_r}\right)e^{-\Delta_{A_r-x+t}}\right)^2\right\}, \tag{9}$$

so that $VAR\left[{}_{A_r-x}S_x \mid \Delta\right]$ is given by the sum in (8) and (10) below

$$E_{\Delta}\left[VAR_T\left({}_{A_r-x}S_x \mid \Delta\right)\right] = B^2\left({}_{A_r-x}p_x\right)^2 \begin{Bmatrix} \sum_{t=0}^{\infty}\left({}_t p_{A_r}\right) M_{\Delta_{A_r-x+t}}(-2) - \\ \sum_{t=0}^{\infty}\sum_{t'=0}^{\infty}\left({}_t p_{A_r}\right)\left({}_{t'} p_{A_r}\right) M_{\Delta_{A_r-x+t}+\Delta_{A_r-x+t'}}(-1) \end{Bmatrix}. \quad (10)$$

So far we have not assumed any specific probabilistic structure for the interest rate, and no form for the survival distribution which we shall compute using life tables. We now specialize our results to the case of autoregressive series of order 2, *AR(2)* for the force of interest series Δ. As we discovered both in the cases of Israeli short-rate data, and US short rate data, *AR(1)* processes fit to the force of interest data led to fitted process parameters that upon simulation yielded a large fraction of negative interest rates. However, *AR(2)* processes led to very few negative rates that could be turned into zero without substantially altering the process. Specifically, we fit the process

$$\Delta_t = \theta + \phi_1\left(\Delta_{t-1} - \theta\right) + \phi_2\left(\Delta_{t-2} - \theta\right) + \varepsilon_t, \quad (11)$$

where the innovations ε_t are independent normal $\left(0, \sigma^2\right)$ variables, θ is the fixed mean of the process, and ϕ_1 and ϕ_2 are the auto regression parameters of orders 1 and 2 respectively that satisfy the requirements $-1 < \phi_1 < 1$, $\phi_1 - \phi_2 < 1$, $\phi_1 + \phi_2 < 1$, insuring that the series is stationary. For this process, the moment generating function is Δ_t, at any fixed time t, is given by

$$M_{\Delta_t}(u) = e^{t\theta u + u^2\sigma^2 G(u)}, \quad (12)$$

with

$$G(x) = \lambda G_1(x) + (1-\lambda)G_2(x),$$

$$G_i(x) = \frac{x}{2}\left(\frac{1+\psi_i}{1-\psi_i}\right) - \psi_i\left(\frac{1-\psi_i^{\,2}}{\left(1-\psi_i\right)^2}\right),$$

where ψ_1 and ψ_2 are reciprocals of the roots of the autoregressive process characteristic equation

$$\phi(r) = (1-\phi_1)r - \phi_2 r^2 = 0, \quad (13)$$

(See Panjer and Bellhouse (Panjer and Bellhouse, 1980) and Parker (Parker, 1994) and

$$\lambda = \frac{\psi_1(1-\psi_2^2)}{\left[(\psi_1 - \psi_2)(1+\psi_1\psi_2)\right]}, \quad (14)$$

where ϕ_1 and ϕ_2 are the auto regression parameters. Since we can also write

$$M_{\Delta_s+\Delta_t}(u) = e^{(s+t)\theta u + 2u^2\sigma^2\left(G(s)+G(t)\right) - u^2\sigma^2\left(G(|t-s|)\right)}, \quad (15)$$

we have an explicit formula for $E[{}_{A_r-x}S_x]$ and $VAR[{}_{A_r-x}S_x]$ in the normal autoregressive *AR(2)* case as well. We emphasize again that in these models volatility is assumed to be fixed.

2.2. Options on Traditional Pensions – The Stochastic Discrete-Time Case

The price of a European call option on a discounted pension annuity for a particular realization, or sample path of Δ is

$$C = \left[{}_{A_r-x}S_x - Ke^{-\Delta_{A_r-x}}\right]^+, \quad (16)$$

where T denotes the random time of death, which is known to exceed x, $A_r > x$, K is the strike price of the option, and $a^+ = \max(0, a)$. Recall that $T = x$ at the time of purchase of the call option, and $T = A_r$ at retirement, the exercise date of the option. Also t when $T = x$. Using the usual conditional expectation

$$E(C \mid \Delta, T > x) = \left({}_{A_r - x}p_x\right) \sum_{t = A_r + 1}^{\infty} P[T = t \mid T > A_r] \left[\sum_{k=0}^{t - A_r} B e^{-\Delta_{A_r - x + k}} - K^{-\Delta_{A_r - x}} \right]^{+}, \tag{17}$$

and

$$E(C \mid T > x) = \sum_{t = A_r + 1}^{\infty} P[T = t \mid T > x] E_{\Delta} \left[\sum_{k=0}^{t - A_r} B e^{-\Delta_{A_r - x + k}} - K e^{-\Delta_{A_r - x}} \right]^{+}. \tag{18}$$

We do not further specialize this formula to the *AR(2)* Gaussian case because the heavyside (plus) function prevents us from writing this expression for the conditional expectation of C, in terms of known quantities like Gaussian moment generating functions. This formula is however easily amenable to simulation.

2.3. Simulation Studies: The Discrete Constant Variance Case

Consider the case of insured individuals whose ages are as specified in the table below and who are interested in buying a European call option on their pension annuity of $2000 per month which will be paid from their retirement to their death. Assume that age of retirement is $A_r = 65$, and the force of interest follows a constant variance *AR(2)* autoregressive model with parameters $\phi_1 = 1.43586$ and $\phi_2 = -0.47069$ and $\sigma = 0.007$ per year. These parameters were obtained upon fitting an *AR(2)* model to the force of interest to the monthly Israeli nominal bank-rate interest series from 1/1985 to 12/2002. This rate incorporates the appropriate market risk premium relevant to the cash flows we price. The observed origin of this parameterization helped insure that few negative interest rates were produced in the simulation, and few interest rates larger than 1 were obtained in the simulation. The parameter θ from (11) was chosen to correspond to 3%, 5%, and 10% annual rate respectively. The single mean premium for the pension plan, for the three average rates, and the three ages at purchase time, are given in Table 1. The results reported in Table 1 may be compared with the corresponding values under fixed interest rate of 3%, 5%, and 10% annual rate, respectively.

For example, the mean price of the pension plan, at 10% interest rate, for an insured aged 40, is $11,421($3,323), whereas the price of the same plan for that individual at a fixed rate of 10% is $13,552; Similarly, at age 50, with the same mean rate, the mean price was $31,511($8,172), and the fixed rate price was $33,880. The values in parentheses refer to the simulation standard deviations, moderating some of the discrepancy observed in the mean stochastic price and the fixed rate price.

In this simulation study we actually simulated the force of interest series only, as we used actual life tables that gave us the conditional expectation of the single premium given the force of interest series explicitly. The simulation standard deviations we report in parentheses under the mean premia in Table 1, are in fact estimates of the conditional standard deviation of the premium given the force of interest series. All simulation results reported in this paper represent the outcome of 10,000 simulation runs from the sample path of the interest rates series.

In the same simulation we also computed the mean price of the European option as a function of the mean force of interest, the age of the insured at the time of purchase of the option and the strike price, when the option is exercised at retirement time. The results are then compared to the corresponding values under non-stochastic interest rates. Again, the expected prices under the time-invariant interest rate and the stochastic *AR(2)* force of interest series with fixed variance are strikingly similar. The simulation standard deviations reported are again estimates of the conditional standard deviations of the option price given the interest rate series.

Table 1

The mean single premium by age at purchase, and by annual interest rate with an *AR(2)* constant volatility model for the interest rate*

Age *x* \ Mean *r*	10%	5%	3%
30	$4,312 ($1,415)	$34,048 ($11,644)	$79,821 ($27,833)
40	$11,421 ($3,323)	$54,870.9 ($16,811)	$105,390 ($33,052)
50	$31,511 ($8,172)	$92,091 ($25,294)	$144,889 ($40,795)

* The number in parentheses represents the simulation standard deviation for the corresponding parameter. The autoregression parameters are $\phi_1 = 1.43586$, $\phi_2 = -0.47069$ and $\sigma = 0.007$.

Table 2

Mean cost and standard deviation of a European call option by strike prices, age at purchase, and mean interest rates**

Age *x*	K	E(C)	E(C)	E(C)
mean rate		$r = 10\%$	$r = 5\%$	$r = 3\%$
30	$30,000 $40,000 $50,000	$3,506 ($1,028) $3,258 ($966) $3,013 ($906)	$29,455 ($8,754) $28,041 ($8,404) $26,643 ($8,062)	$70,564 ($21,125) $67,720 ($20,427) $64,910 ($19,742)
40	$30,000 $40,000 $50,000	$9,325 ($2,291) $8,668 ($2,161) $8,020 ($2,034)	$47,691 ($11,921) $45,411 ($11,477) $43,160 ($11,043)	$93,636 ($23,617) $89,882 ($22,892) $86,172 ($22,183)
50	$30,000 $40,000 $50,000	$25,442 ($4,967) $23,630 ($4,683) $21,842 ($4,409)	$79,044 ($15,742) $75,218 ($15,158) $71,439 ($14,590)	$127,066 ($25,588) $121,903 ($24,810) $116,801 ($24,054)

** The number in parentheses represents the simulation standard deviation for the corresponding parameter. The autoregression parameters are $\phi_1 = 1.43586$, $\phi_2 = -0.47069$ and $\sigma = 0.0005$.

The results reported in Table 2 may again be compared to the corresponding values under fixed interest rate of 3%, 5%, and 10% annual rate, respectively.

For example, the mean price of the pension plan, at 5% interest rate, for an insured age 30, with a strike price of $30,000 and $40,000 are $29,455($8,754) and $28,041($8,404) respectively. The corresponding values for fixed interest rate were $29,995 and $28,455. Whereas the price of the same plan for that individual at a fixed rate of 10% is $13,552. Similarly, at age 50, with the same mean rate, the mean price was $31,511($8,172), and the fixed rate price was $33,880. Again we note that these results may be partially explained by price volatility in the stochastic case, and the upward trend due to interest rate volatility is again noticeable.

We remark that the formulae we presented for the expected value of a single premium, and the expected cost of a single option on the pension benefit, can be extended to the dynamic survival case where the survival distribution depends on time. To visualize the situation, consider the discrete-time stochastic process of survival that starts at time $t = 0$ at state 0 (alive), and transitions into state 1 (dead), at some time T. State 1 is absorbing for this survival process. We assume the process to be Markovian. The n-step transition probability for this process $P[X(t+n) = 1 \mid X(t) = 0]$ may depend on calendar time t, making the process a non-homogeneous Markov process. Written differently,

$P_t[X(t+n)=1 \mid X(t)=0]=\dfrac{P_t[T \geq t+n]}{P_t[T \geq t]}$. Thus in formulae (5) and (18) we need to replace ${}_{(A_r-x)}p_x$ by $\dfrac{P_x[T \geq A_r]}{P_x[T \geq x]}$ and ${}_t p_{A_r}$ by $\dfrac{P_{A_r}[T \geq A_r+t]}{P_{A_r}[T \geq A_r]}$.

2.4. Simulation Studies: The Discrete Constant Variance Case -AR(2)

In order to study the effect of larger volatility on the single premium of a pension annuity plan, and the price of a European option on it, we simulated homoscedastic series with the same autoregressive parameters as before, but with different standard deviations. We were seriously limited in varying σ because the simulated series tended to explode when the standard deviation was substantially larger than the empirical one fitted to real data. Because the standard deviations thus remained in a fairly short interval, we report only the results for the smallest and largest σ we used $\sigma_1 = 0.0012$ and $\sigma_2 = 0.006$.

Table 3

The mean single premium by age at purchase, and mean annual interest rate for an *AR(2)* constant volatility rate model for two values of σ

Age x \ Mean rate r	10%	5%	3%
30 σ_1 / σ_2	\$4,096 (\$265) \$4,140 (\$1,364)	\$32,234 (\$2,174) \$32,698 (\$11,241)	\$75,435 (\$5,186) \$76,663 (\$26,896)
40 σ_1 / σ_2	\$11,199 (\$659) \$11,288 (\$3,428)	\$53,615 (\$3,312) \$54,241 (\$17,328)	\$102,795 (\$6,496) \$104,190 (\$34,095)
50 σ_1 / σ_2	\$31,208 (\$1,519) \$31,468 (\$7,713)	\$90,900 (\$4,686) \$91,922 (\$23,893)	\$142,781 (\$7,548) \$144,585 (\$38,575)

The mean premium was only slightly raised due to an increased standard deviation. Note the almost linear effect of the rate process standard deviation on the premium simulation standard deviation. When the former was multiplied by 5, the latter was also approximately multiplied by 5. This linear relationship is not exact, but held in our simulations for the intermediate standard deviations as well.

Using the same parameters, and the same simulation runs, we also estimated the mean price of a European option on the pension contracts priced in Table 3. Again we may compare the results with constant interest rate, and under two volatility values, under the same set of conditions on age at purchase, mean interest rate, exercise time, and strike price. Remarkably, the mean price of the option increased slightly with a higher rate process standard deviation, and its standard deviation was approximately proportional to the process standard deviation.

Table 4

Mean price and standard deviation of a European call option by strike price and mean

Age x	K, σ_1, σ_2	E(C)	E(C)	E(C)
		$r = 10\%$	$r = 5\%$	$r = 3\%$
30	\$30,000 σ_1 σ_2	\$3,304 (\$194) \$3,321 (\$984)	\$27,675 (\$1,650) \$27,858 (\$8,350)	\$66,207 (\$3,971) \$66,701 (\$20,122)
	\$40,000 σ_1 σ_2	\$3,067 (\$182) \$3,082 (\$921)	\$26,321 (\$1,580) \$26,498 (\$7,997)	\$63,485 (\$3,832) \$63,966 (\$19,418)
	\$50,000 σ_1 σ_2	\$2,833 (\$170) \$2,847 (\$860)	\$24,984 (\$1,512) \$25,154 (\$7,652)	\$60,795 (\$3,695) \$61,264 (\$18,727)
40	\$30,000 σ_1 σ_2	\$9,034 (\$468) \$9,050(\$2,376)	\$46,024 (\$2,408) \$46,131 (\$12,186)	\$90,194 (\$4,742) \$90,430 (\$23,987)
	\$40,000 σ_1 σ_2	\$8,385 (\$439) \$8,400 (\$2,228)	\$43,772 (\$2,309) \$43,876 (\$11,680)	\$86,485 (\$4,581) \$86,716 (\$23,162)
	\$50,000 σ_1 σ_2	\$7,745 (\$411) \$7,759 (\$2,083)	\$41,548 (\$2,212) \$41,649 (\$11,186)	\$82,820 (\$4,423) \$83,046 (\$22,352)
50	\$30,000 σ_1 σ_2	\$25,168 (\$1,049) \$25,236 (\$5,343)	\$77,991 (\$3,329) \$78,196 (\$16,949)	\$125,204 (\$5,407) \$125,531 (\$27,524)
	\$40,000 σ_1 σ_2	\$23,359 (\$996) \$23,425 (\$5,077)	\$74,173 (\$3,220) \$74,373 (\$16,402)	\$120,052 (\$5,262) \$120,372 (\$26,794)
	\$50,000 σ_1 σ_2	\$21,575 (\$945) \$21,639 (\$4,821)	\$70,402 (\$3,114) \$70,598 (\$15,869)	\$114,961 (\$5,120) \$115,275 (\$26,083)

* The number in parentheses represents the simulation standard deviation for the corresponding parameter.

2.5. Pension Option Prices as Function of Strike Price and Arbitrary Exercise Date

We now allow purchasers of annuity pension options to exercise their options contract before retirement. Thus exercise time for pension options may be anywhere between purchase time and A_r, the age of retirement. The mean option price for strike time T_s, $x \le T_s \le A_r$, is then

$$E(C \mid T > x) = \sum_{t=A_r+1}^{\infty} P[T = t \mid T > x] E_{\Delta}\left[\sum_{k=0}^{t-A_r} Be^{-\Delta_{A_r - x + k}} - Ke^{-\Delta_{A_r + T_s - x}}\right]^{+} \tag{19}$$

Again, this formula is completely general, and does not depend on any particular model for interest rates. It is strictly a computational formula, and is not particularly useful for theoretical evaluations. We present our simulation results for various exercise times in Table 5. A quick perusal through the table reveals that as expected, as the exercise age decreases from the age of retirement to the age at purchase, the mean price and its standard deviation also decrease. Other table results require few comments. Obviously, when exercise age is smaller than age at purchase, the premium and the option price are null. As expected both the mean premium and the mean price and the corresponding standard deviations are highly affected by the exercise date, in the predictable direction.

Table 5

Mean price and standard deviation of the price, mean interest rate and exercise date for an *AR(2)* interest rate*

Age x	K, Exercise Age	E(C)	E(C)	E(C)
		$r = 10\%$	$r = 5\%$	$r = 3\%$
30	$30,000 45 55	$230 ($372) $2,290 ($891)	$22,422 ($8,696) $26,588 ($8,687)	$63,320 ($21,804) $67,091 ($21,398)
	$40,000 45 55	$21 ($102) $1,703 ($789)	$19,025 ($8,252) $24,368 ($8,261)	$58,462 ($21,196) $63,378 ($20,670)
	$50,000 45 55	$1.6 ($16) $1,172 ($688)	$15,834 ($7,786) $22,202 ($7,854)	$53,739 ($20,594) $59,730 ($19,965)
40	$30,000 45 55	$676 ($1,263) $6,165 ($2,394)	$36,778 ($14,263) $43,681 ($13,552)	$85,210 ($28,318) $90,351 ($27,063)
	$40,000 45 55	$98 ($461) $4,579 ($2,174)	$31,184 ($13,850) $40,021 ($13,046)	$78,642 ($27,952) $85,337 ($26,362)
	$50,000 45 55	$12.5 ($107) $3,147 ($1,935)	$25,940 ($13,332) $36,455 ($12,555)	$72,259 ($27,550) $80,412 ($25,677)
50	$30,000 45 55	$0 ($0) $16,664 ($6,176)	$0 ($0) $72,394 ($19,930)	$0 ($0) $122,866 ($32,245)
	$40,000 45 55	$0 ($0) $12,299 ($5,842)	$0 ($0) $66,243 ($19,599)	$0 ($0) $115,958 ($31,885)
	$50,000 45 55	$0 ($0) $8,382 ($5,390)	$0 ($0) $60,252 ($19,253)	$0 ($0) $109,176 ($31,518)

* The number in parentheses represents simulation standard deviation, e.g. means, exercise date.

3. Stochastic Interest Rates: The Case of Continuous Time

3.1. Continuous Models: Some History

Financial contracts that depend on levels of interest rates, with zero-coupon discount bonds being the prime example, but also including options on pension annuity contracts, have attracted much attention in the financial literature in recent years. Pricing these contracts required the formulation of adequate stochastic models for short-term interest rates in continuous time. Recent models for the short rate process, or the forward-rate process, may be divided into two groups that are distinguished primarily by the way their volatility is modeled. Their basic structure is that of a continuous autoregression, usually of order 1, [see, e.g., Koedijk, Nissen, Schotman, and Wolff (Koedijk, Nissen, Schotman, and Wolff , 1997), and the many references therein]. Older models, such as Merton's (Merton , 1973) and Vasicek's (Vasicek, 1977), derived via an argument of a simple economic equilibrium, feature a fixed volatility. Cox, Ingersoll, and Ross (CIR, 1985), derived the first stochastic volatility model from first economic principles. Their is a continuous time autoregressive model with conditional volatility, given the 'past' of the process up to time t^{-}, just prior to time t, that is the square root of the short rate at time t^{-}. Chan, Karolyie et al. (op. cit.) have generalized the CIR model into a family of continuous autoregressive models with stochastic volatility, that include many of the important models in use up to that time. The family is specified by the stochastic differential equation

$$dr_t = \kappa(\theta - r_t)dt + \sigma_t r_t^{\gamma} dB_t\,, \tag{20}$$

where κ, θ, and $\gamma \geq 0$, and B_t is a standard Brownian Motion. The stochastic volatility σ_t is adapted to the B_t process (essentially determined at time t by its history up to time t^{-}), where the conditional volatility is $V_t = \sigma_t * r_{t^-}^{\gamma}$. The γ parameter in this model has been dubbed the elasticity parameter because it controls the distance the process reaches in the positive half-space, and the frequency of its 'waves'. The CIR process features $\gamma = 0.5$. Chan, Karolyie et al. (op. cit.) have found the value $\gamma = 1.5$ to best fit their data of empirical monthly interest-rates of US T-bills. Subsequent studies have disputed this finding, suggesting that the high value of the elasticity parameter gamma was in fact a result of model misspecification that did not take into account the new regime imposed by the Federal Reserve on interest rates [see Koutmos (Koutmos, 1998) and references therein], and in fact the CIR model, featuring $\gamma = 0.5$ fits the data locally far better. In forward rate simulation of these continuous models, the Euler approximating discrete model

$$r_t - r_{t-1} = \kappa(\theta - r_{t-1}) + \sigma_{t-1} r_{t-1}^{\gamma} Z_{t-1} \text{ for } t = \ldots,-2,-1,0,1,2,\ldots, \tag{21}$$

where Z_t represents an independent, identically distributed sequence of *Normal* (0, 1) deviates, is used. More recent empirical studies of short-term interest rates (See Koutmos (Koutmos, 1998, 2000) and Bali (Bali, 2003)) suggest that discrete time approximations to such continuous processes by *AR(1)* or *AR(2)* models with *ARCH* or *GARCH* terms for the conditional volatility given the past of the interest rate process, fit much empirical data far better. The latter processes are reported to account for prolonged walks observed in empirical interest rate series in the positive half-space, without causing frequent negative rates. The simplest instance of a linear *ARCH-GARCH* (discrete time) model is an autoregressive model *AR(k)*

$$r_t - \theta = \psi_0 + \sum_{j=1}^{k} \psi_j \cdot (r_{t-j} - \theta) + \xi_t\,, \tag{22}$$

where the errors $\xi_t = V_t^{0.5} Z_t$ and the Z_t are independent identically distributed standard normal variates and the volatility process V_t, in the *GARCH* (p, q) specification, is given by the autoregression

$$V_t = \alpha_0 + \sum_{j=1}^{p} \alpha_j * \xi_{t-j}^2 + \sum_{j=1}^{q} \beta_j * V_{t-j}, \tag{23}$$

with fixed parameter vectors α and β.

More recent volatility models that are linear combinations of elasticity terms, and *ARCH-GARCH* terms, have also been proposed by Bali (op. cit.). He also proposed two-factor models which describe the interest rate process using a continuous autoregression with errors given by one Brownian Motion, and a conditional volatility model, also an autoregression, with another Brownian Motion, independent of the first. Although these two-factor models seem promising, we do not pursue them in this work.

3.2. Pricing Traditional Pension Annuity Plans and European Options on Annuity Pension Plans

In our modeling of the dynamic forward interest rate, or interest force, in continuous time, and fitting it to empirical data, we have, of course, used the discrete Euler approximation described in equation (21). The empirical data we used for fitting a continuous model, via its discrete approximation, are the monthly Eurodollar annualized deposit interest rate series, from 6/1953 to 6/2003, published by Federal Reserve in its H.15 database. We then followed the forward interest rate model fitting by a simulation using the parameters we have estimated, and eventually computed the single premium for a pension plan, and the price of a European option on this pension plan.

Because we used discrete approximations to the continuous process, the main difference in the continuous time case from the discrete case is that in the latter we used stochastic volatility models, whereas in the former we used fixed volatility models. As we saw earlier the fixed volatility models permitted the study of the dependence of option prices on important parameters such as variance size, strike time, and age at purchase of option. The stochastic volatility models produce chaotic volatility that prohibits the study of these parameters, because small changes in parameter values lead to the explosion of the series of rates.

We have tried to fit combination models to all of these possible models, and found, using maximum likelihood, the *ARCH-GARCH* volatility models to fit our empirical data best, and then lead to more successful simulations that produce reasonable series of force of interest, and then of interest, that do not hit below zero, or above one too often.

The formulae we obtained in the discrete stochastic case can easily be modified to fit the continuous stochastic forward interest rate case. We use the same notation for the continuous force of interest, and cumulative force of interest, and allow them to follow an unspecified stochastic model. We assume that the continuous stochastic survival (time of death) variable T, which follows an unspecified continuous distribution, is independent of the stochastic force of interest process δ_t. The single premium paid by an insured aged x to purchase an annuity insurance that would pay B from age $A_r = 65$, say, till his death, provided he is alive at age A_r, is random and is given by the integral:

$${}_{A_r - x}S_x = B \int_{A_r}^{T} \exp[-\Delta(t-x)]dt, \tag{24}$$

where $\Delta(t)$ denotes the cumulative stochastic force of interest at time t, and $A_r > x$. The complete continuous time series will again be denoted by Δ. Here

$$\delta(t) = \log(1 + r(t)), \tag{25}$$

where $r(t)$ is the stochastic forward interest rate, and the cumulative force of interest is given by

$$\Delta(t) = \int_0^t \delta(u)du. \tag{26}$$

B is the fixed benefit paid per unit time from retirement A_r time onward, till the death time T of the insured. By the independence between T and $\delta(t)$ we can write the expectation of ${}_{A_r - x}S_x$ as

$$E[{}_{A_r-x}S_x] = E_\Delta[E_T({}_{A_r-x}S_x \mid \Delta)] \tag{27}$$
$$= \left[B(1 - F_T(A_r))/(1 - F_T(x))\right] \int_{t=0}^{\infty} \int_{u=0}^{t} E_\Delta\left(e^{-\Delta(A_r - x + u)}\right) du f_T(t)dt,$$

where F_T denotes the cumulative distribution function of T. This expectation may also be expressed in terms of the moment generation function $M_{\Delta(t)}(u)$ of $\Delta(t)$ at u. For Gaussian processes the inner expectations, being the moment generating function of a Gaussian variable, take on a simple exponential form, and depending on the density $f_T(\cdot)$ the integral may be computed explicitly. In any case Monte Carlo simulation is straight forward. Furthermore, we can write again

$$VAR\left[{}_{A_r-x}S_x\right] = VAR_\Delta\left[E_T\left({}_{A_r-x}S_x \mid \Delta\right)\right] + E_\Delta\left[VAR_T\left({}_{A_r-x}S_x \mid \Delta\right)\right] \tag{28}$$

and express it in terms of joint and marginal moment generating functions, as in the discrete case.

The pricing formula for the continuous time case remains formula (16) but its expectation becomes

$$E(C \mid T > x) = \int_{A_r}^{\infty} (f_T(t) \mid P[T > x]) E_\Delta\left[\int_0^{t-A_r} Be^{-\Delta_{A_r - x + u}} du - Ke^{-\Delta_{A_r - x}}\right]^+ dt. \tag{29}$$

Note that $\frac{f_T(t)}{P[T > x]}$ for $t > x$ is not a hazard rate, in contrast to Milevsky and Promislow's (op. cit.) formula, and is also non-stochastic. However, if we assume that the distribution of T is stochastic, that is survival changes dynamically with time, the only change we would have to make in the last formula is to add an expectation with respect to the distribution of $\frac{f_T(t)}{P[T > x]}$. We also note that unlike Milevsky and Promislow (op. cit.), we have not carried out our computations under the Martingale measure. The computation under the Martingale measure simplifies the formula substantially, by moving the expectation under the inner integral sign. The formula as it stands serves however very well for simulation purposes. Once we choose a model for the force of interest $\delta(\cdot)$ via a stochastic differential equation, we replace the latter by its discrete approximation, which we then use with identical time intervals (e.g., months) in both the life tables and the force of interest discrete process.

3.3. Simulations in the Continuous Time Case

We first fit several models to the Federal Reserve monthly interest data. The best fit was obtained by an *ARCH-GARCH AR(2)* model of order *(p = 1, q = 1)*, with *ARCH* parameter $\alpha = 0.4014$, *GARCH* parameter $\beta = 0.4998$, $\alpha_0 = 0.10537 * 10^{-7}$, mean $\theta = 0.1927 * 10^{-3}$, $\psi_1 = -1.426$, $\psi_2 = 0.470$, and $\sigma = 0.406 * 10^{-3}$. These estimated parameters were then used in a simulation to compute the mean single premium of a plan (Table

6) and the price of the plan (Table 7) as a function of age at purchase, and mean rate. To reduce the size of these tables, we assume that the option is exercised at retirement time, and included only three strike amounts.

Although the results reported in the two tables below cannot be directly compared to the corresponding tables for time-invariant volatility, one can nonetheless note the remarkably small standard deviation relative to the mean amount, for both the premium and the option price in the stochastic volatility case.

Table 6

The mean single premium by age at purchase and mean annual interest rate for *ARCH – GARCH (1,1), AR(2)* interest rate model*

Age *x* \ Rate *r*	10%	5%	3%
30	$4,550 ($433)	$37,320 ($3,732)	$89,390 ($9,148)
40	$12,502 ($1,097)	$62,383 ($5,808)	$122,419 ($11,718)
50	$35,001 ($2,564)	$106,250 ($8,339)	$170,813 ($13,823)

* The number in parentheses represents the simulation standard deviation for the corresponding parameter.

Table 7

Mean cost and standard deviation of a European call option by strike prices and mean interest rates for *ARCH-GARCH (1,1), AR(2)* interest rate model*

Age *x*	K	E(C)	E(C)	E(C)
Mean rate		$r = 10\%$	$r = 5\%$	$r = 3\%$
30	$30,000 $40,000 $50,000	$3,723 ($291) $3,478 ($273) $3,236 ($256)	$32,484 (2,519) $31,091 ($2,420) $29,711 ($2,322)	$79,501 ($6,141) $76,702 ($5,944) $73,927 ($5,750)
40	$30,000 $40,000 $50,000	$10,258 ($723) $9,587 ($681) $8,923 ($641)	$54,469 ($3,871) $52,142 ($3,731) $49,835 ($3,593)	$109,223 ($7,793) $105,389 ($7,565) $101,589 ($7,341)
50	$30,000 $40,000 $50,000	$28,645 ($1,773) $26,765 ($1,689) $24,905 ($1,607)	$92,543 ($5,793) $88,578 ($5,620) $84,648 ($5,451)	$152,042 ($9,583) $146,692 ($9,353) $141,389 ($9,127)

* The number in parentheses represents the simulation standard deviation for the corresponding parameter.

4. Conclusions

In this paper we analyzed the behavior of a recently introduced pension insurance instrument, a European call option defined on pension annuity. Under this plan, insured parties can buy an option on their pension annuity benefit, granting them the opportunity to buy their discounted annual pension annuity benefit prior to or at the age of retirement from the options writers, at a defined strike price. The analysis was carried out under stochastic interest rates. We considered a variety of different stochastic models for interest rate, and in all models we found that as the standard. The use of this European call option for pensions is a new method which enables individuals to subscribe to a pension annuity at a later age, fixing the terms of payment in advance, while the current value paid by the individual is somewhat higher than standard pension annuity.

Because Milevsky & Promislow's (Milevsky & Promislow, 2001) work treated the related problem of valuing European-style options on the mortality-contingent claim that pays one lump sum upon surviving a pre-specified period, we review our contribution relative to their study. The insurance product they treated is also known as an endowment policy, in contrast to traditional life annuity, and is quite similar to a zero-coupon bond. Milevsky and Promislow go on to regard, under certain ordering conditions, an option on a traditional pension plan as a basket of such endowment plans of different maturities. In the discrete time case only a one year horizon is actually worked out. Their argument for replacing the pension annuity by a basket of options appears to suggest that the state space of interest rates must also be discrete. In the continuous time case, Milevsky and Promislow adopt Duffie and Singleton's (Duffie and Singleton, 1997) approach to the valuing of European options on defaultable bonds. Here Milevsky and Promislow replace the default hazard by the insured mortality hazard. For the single endowment plan they posit a Cox, Ingersoll, and Ross (op. cit.) model for the stochastic interest rate, and an independent Brownian motion with a linear trend, for the logarithm of the stochastic hazard rate. No extension is actually given for an option on a pension plan in continuous time. A small simulation study computes the price of a basket of options on endowment policies as a function of their fixed volatility of their interest rate process. Although a formula is developed for the value of an endowment option in terms of the Matingale measure that turns the interest rate process into a martingale, the simulation appears to simply use the CIR model under the usual measure, without making use of the formula.

The main difference between our approach and that of Milevsky and Promislow (op. cit.) is in the fact that we give general formulae for the valuation of European options on pension plans directly, without assuming any particular form for either the survival distribution or the interest rate process. Both approaches assume independence between survival and interest rate. In the discrete case we also provide variance formulae for the (stochastic) price actually paid by the insured who buys an option at some time before retirement. Similar formulae are also possible in the continuous time case.

References

1. T.G. Bali. Modeling the Stochastic Behavior of Short-Term Interest rates: Pricing Implications for Discount Bonds. // The Journal of Banking and Finance, 2003. Volume 27, – № 2. – pp. 201-228.
2. Ballotta, L., S. Haberman. Valuation of Guaranteed Annuity Conversion Options // Insurance, Mathematics and Economics, 2003. – № 33. – pp. 87-108.
3. Boyle, P., M. Hardy. Guaranteed Annuity Options // Working paper, 2003. - University of Waterloo, Canada.
4. Cairns. A Family of Models for the Term-Structure of Interest Rates: An Application to Guaranteed Annuity Options // Presentation at the IME 2002 Conference, Lisbon.
5. Chan, K.C., Karolyi, G.A., Longstaff, F.A., and A.B. Sanders. An Empirical Comparison of Alternative Models of the Short-Term Interest Rate // The Journal of Finance, 1992. Volume 47, – № 3. – pp. 1209-1227.
6. Cox, C.J., Ingersoll, J.E., S.A. Ross. A Theory of the Term Structure of Interest Rates // Econometrica, 1985. Volume 53, – № 2. – pp. 385-407.
7. Duffie, D., K. Singleton. An Econometric Model of the Term Structure of Interest Rate Swap Yields // The Journal of Finance, 1997. Volume 52, – № 4. – pp. 1287-1321.
8. Koedijk, K.G., Nissen, F.G.J.A., Schotman, P.C., C.C.P., Wolff. (1997). The dynamics of Short-term Interest Rate Volatility reconsidered // European Finance Review, 1997. – № 1. – pp. 105-130.
9. G. Koutmos. The Volatility of Interest Rates Across Maturities and Frequencies // The Journal of Fixed Income, 1998. December. – pp. 27-31.
10. G. Koutmos. Modeling Short-Term Interest Rate Volatility: Information Shocks Versus Interest Rate Levels // The Journal of Fixed Income, 2000. March. – pp. 19-26.

11. P. Lee. Pricing and Reserving for guaranteed Annuity Option // Presentation for Den Danske Aktuarforening, 2001. http://www.inqa.com/conferences.htm.
12. R.C. Merton. Theory of Rational Option Pricing // Bell Journal of Economics and Management Science, 1973. – № 4, – pp. 141-183.
13. Milevsky, M.A., D.A. Promislow. Mortality Derivatives and Option to Annuities // Insurance: Mathematics and Economics, 2001. Volume 29. – pp. 299-318.
14. Nielsen, A.J., K. Sandmann. Pricing of Asian Exchange Rate Options Under Stochastic Interest Rates as a Sum of Options // Finance and Stochastics, 2002. Volume 6. – pp. 355-370.
15. Panjer, H.H., D.R. Bellhouse. Stochastic Modelling of Interest Rates with Applications to Life Contingencies // Journal of Risk and Insurance, 1980. Volume 47. – pp. 91-110.
16. G. Parker. Two Stochastic Approaches for Discounting Actuarial Functions // Astin Bulletin, 1994. Volume 24, – № 2. – pp. 167-181.
17. Pelsser. Pricing and Hedging Guaranteed Annuity Options via Static option Replication. Insurance: Mathematics and Economics, 2003. Volume 33. - pp.283-296.
18. S. Ross. Introduction to Probability Models. – New York: Academic Press, Inc., 1997.
19. O. Vasicek. An Equilibrium Characterization of the Term Structure // Journal of Financial Economics, 1977. Volume 5. – pp. 177-188.
20. Wilkie, D., Waters, H., S. Yang. Reserving Pricing and Hedging for Policies with Guaranteed Annuity Options // British Actuarial Journal, 2003. To appear.
21. Yosef, R., Benzion, U., and S.T. Gross. Pricing of a European Call Option on Pension Annuity Insurance // The Journal of Insurance Issue, 2003. Forthcoming, Spring 2004.