





# “Estimating the value-at-risk of JSE indices and South African exchange rate with Generalized Pareto and stable distributions”

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# ESTIMATING THE VALUE-AT-RISK OF JSE INDICES AND SOUTH AFRICAN EXCHANGE RATE WITH GENERALIZED PARETO AND STABLE DISTRIBUTIONS

## Abstract

South Africa's economy has faced many downturns in the previous decade, and to curb the spread of the novel SARS-CoV-2, the lockdown brought South African financial markets to an abrupt halt. Therefore, the implementation of risk mitigation approaches is becoming a matter of urgency in volatile markets in these unprecedented times. In this study, a hybrid generalized autoregressive conditional heteroscedasticity (GARCH)-type model combined with heavy-tailed distributions, namely the Generalized Pareto Distribution (GPD) and the Nolan's S0-parameterization stable distribution (SD), were fitted to the returns of three FTSE/JSE indices, namely All Share Index (ALSI), Banks Index and Mining Index, as well as the daily closing prices of the US dollar against the South African rand exchange rate (USD/ZAR exchange rate). VaR values were estimated and back-tested using the Kupiec likelihood ratio test. The results of this study show that for FTSE/JSE ALSI returns, the hybrid exponential GARCH (1,1) model with SD model (EGARCH(1,1)-SD) outperforms the GARCH-GPD model at the 2.5% VaR level. At VaR levels of 95% and 97.5%, the fitted GARCH (1,1)-SD model for FTSE/JSE Banks Index returns performs better than the GARCH (1,1)-GPD. The fitted GARCH (1,1)-SD model for FTSE/JSE Mining Index returns is better than the GARCH (1,1)-GPD at 5% and 97.5% VaR levels. Thus, this study suggests that the GARCH (1,1)-SD model is a good alternative to the VaR robust model for modeling financial returns. This study provides salient results for persons interested in reducing losses or obtaining a better understanding of the South African financial industry.

## Keywords

GARCH-type models, Generalized Pareto Distribution, backtesting, Nolan's S0-parameterization-stable distributions, value-at-risk

## JEL Classification

C22, C58, G32

## INTRODUCTION

In the recent past, the South African financial industry has experienced a whirlwind of challenges due to several isolated disruptive global or domestic events, the most recent being the COVID-19 pandemic. The South African government, like many governments across the world, declared a national state of disaster, and such decisions caused detrimental impacts on the global economy. Stunted growth, market disruptions and consumer uncertainty are some of the numerous negative effects that have arisen as a result. Nel et al. (2020) stress that financial institutions should remain hyper-vigilant and impose proper risk management and mitigation strategies for business continuity.

In these circumstances, understanding the volatile characteristics of the financial market becomes of key interest in formulating tactical strategies to cope with unexpected disadvantageous events.

Appropriate and reliable models that capture the stylized properties of financial data are recommended. Literature suggests dependency on the generalized autoregressive conditional heteroscedasticity (GARCH) framework in dealing with observed volatility in finance (Jafari et al., 2007). However, even with the success of GARCH models, Mzamane et al. (2013) point out the need for more flexible volatility modelling that caters for the asymmetric effect in financial returns.

To examine this, a Value-at-Risk (VaR) approach with a hybrid GARCH-type model with a heavy tailed distribution extension is investigated for the South African financial market where characteristics of fat tails and high market volatility are apparent.

Policymakers, regulators, risk-averse investors, and insurers could leverage if they identify distributions that adequately capture aspects of financial data and impose risk mitigation approaches. Hence, models that monitor volatile indices and exchange rates, and provide remedies that are useful to investors, are necessary, especially since disruptive events and uncertainty are prevalent in the financial industry for the foreseeable future.

In essence, this study explores the heavy-tailed addition, specifically the generalized Pareto distribution (GPD) and stable distribution (SD) to the standard GARCH model as alternatives to capture signs of asymmetry and volatility clustering to three FTSE/JSE market indices and the US Dollar to the South African Rand, as well as acquire further understanding of the South African financial industry.

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## 1. LITERATURE REVIEW

McNeil and Frey (2000) proposed a method for evaluating VaR by using the pseudo-maximum likelihood method to fit GARCH models and Extreme Value Theory (EVT) to analyze the tail of the innovation distribution for the GARCH Model. The data consists of five historical series of log returns, namely: Standard & Poor's Index (January 1960–1993), DAX Index and BMW index (January 1973 – July 1996), US dollar and British pound (1980–1996), and lastly gold prices (January 1980 – December 1997). This study suggests a conditional approach that models the conditional distribution of asset returns, well suited for estimating VaR. Also, residual distributions are often leptokurtic, and the GPD-approximation is preferred as it can account for asymmetry in the tail.

The application of Extreme Value Theory by Byström (2005) to the case of large electricity prices showed an adequate fit by the GPD distribution. Chifurira and Chinhamu (2017) compared the performance of the GARCH-GPD and GARCH-Pearson type-IV models when estimating the VaR for the South African Mining Index returns. The main findings suggest that the GARCH-GPD and GARCH-Pearson type-IV models outperform the Generalized Hyperbolic distributions.

Nolan (2003) analyzes the appropriateness of stable distributions to financial data. This study used a maximum likelihood (ML) method to estimate the stable parameters. The paper studied the monthly exchange rates between the U.S. Dollar (USD) and the Tanzanian Shilling and ranged from January 1975 to September 1997 where it was found that the USD/Tanzanian Shilling exchange rate was subject to more extreme fluctuations. The study has shown the practicability of stable parameter estimation, and model diagnostics show that stable distributions describe financial data well. In cases where the fit is imperfect or unsuitable, this study highlights a better fit than that of the Normal Gaussian model and highlights the fittingness of stable distributions in VaR calculations.

Borak et al. (2005) emphasize the empirical evidence by Fama (1965) and Mandelbrot (1963) that suggests stable distributions as a heavy-tailed alternative. Stable distributions consider asymmetry and fat tails and are valued models in view of extreme events such as the global market crisis or natural calamities. Empirical evidence shows a strong fit of stable laws for the DJIA index and Boeing stock.

Ilupeju (2016) researched daily returns of FTSE/JSE ALSI returns from 20 May 2005 to 31 May 2016, and the results showed that the APARCH

model with PIDV and GPD for the short position and stable distribution innovations in the long position are best when dealing with the return series. The suitability of the APARCH(1,1) model together with heavy-tailed distributions was suggested by Chifurira and Chinhamu (2019) for FTSE/JSE ALSI returns. It was noted that the volatility phenomenon and asymmetry were explained by the APARCH framework, and the heavy-tailed distributions account for the fat tails in returns. APARCH(1,1) with GPD or PIVD innovations have proven to be robust models for estimating VaR.

Dwarika et al. (2021) studied the volatility dynamics and the risk-return relationship in South Africa using a GARCH approach where FTSE/JSE All-Share Index returns were analyzed from 2009 to 2019. Various GARCH models were employed, and results revealed consistent volatility and a positive risk relationship in the South African market. Furthermore, this study suggests that volatility of the financial return series could be captured by the EGARCH(1,1) model, however, the model failed to account for the asymmetry.

There is limited research on modelling South African data – JSE indices and the USD/South African (ZAR) exchange rate to GARCH-type, GPD and stable distributions (SD) to the best of our literature knowledge. Therefore, this study investigates the FTSE/JSE All-Share Index (FTSE/JSE ALSI), FTSE/JSE Banks Index, FTSE/JSE Mining Index, and the United States of American Dollar to the South African Rand exchange rate (USD/ZAR exchange rate) with the goal of attaining robust VaR models for financial returns. Using a GARCH framework, the stylized facts associated with financial returns such as heavy tails, asymmetry, leverage effect and volatility clustering are assessed through the fitting of suitable generalized Pareto distribution (GPD) and stable distribution (SD). The estimated VaR values are backtested to evaluate each model's robustness using the Kupiec likelihood ratio tests.

## 2. METHOD

This section presents the background theory on the GARCH-type models combined with GPD and SDs. VaR and backtesting procedures are also discussed.

### 2.1. GARCH (1,1)

The GARCH (1,1) model presented by Bollerslev (1986) is considered a parsimonious model of conditional variance that fits many economic time series (Embrechts et al., 1999).

The GARCH (1,1) model is defined as follows:

$$\sigma_x^2 = \alpha_0 + \alpha_1 \alpha_{x-1}^2 + \beta_1 \sigma_{x-1}^2, \quad (1)$$

$$a_x = \sigma_x \varepsilon_x, \quad (2)$$

where  $\varepsilon_x \sim N(0,1)$ ,  $\alpha_0 > 0$ ,  $\beta_1 > 0$  and  $\alpha_1 + \beta_1 < 1$ , and the next variance forecast period is a mix of previous period forecasts and the previous period's squared returns.  $\alpha_1$  shows short-run persistence of shocks, and  $\beta_1$  suggests long-run persistence of shocks. A disadvantage of the GARCH (1,1) model is the symmetric nature of the model.

To consider asymmetry in the return series, extensions of the GARCH model are suggested to account for the weaknesses of the GARCH (1,1) model. The EGARCH (Exponential GARCH) and other asymmetric GARCH-type models are applied to capture the stylized characteristics of financial data.  $\sigma_x^2 = \alpha_0 + \alpha_1 \alpha_{x-1}^2 + \beta_1 \sigma_{x-1}^2$ ,

### 2.2. EGARCH (p, q) model

The EGARCH model is defined in the study by Nelson (1991):

$$\begin{aligned} \log \sigma_x^2 = & \alpha_0 + \\ & + \sum_{i=1}^p \alpha_i \left( \phi Z_{x-i} + \gamma \left[ |Z_{x-i}| - E|Z_{x-1}| \right] \right) + \quad (3) \\ & + \sum_{j=1}^q \beta_j \log \sigma_{x-j}^2, \end{aligned}$$

where  $\alpha_x = \sigma_x \varepsilon_x$  and  $\varepsilon_x \sim N(0,1)$ ,  $\alpha_0$ ,  $\alpha_i$ ,  $\beta_j$ ,  $\phi$  and  $\gamma \in$ .

To guarantee the conditional variances are positive, there are no constraints placed on the  $\alpha_i$  and  $\beta_j$  parameters unlike the GARCH ( $p,q$ ) model. EGARCH( $p,q$ ) models are favored when the negative and positive shocks of the same magnitude have no equal contributing effect to volatility.

Volatility in EGARCH( $p,q$ ) models is calculated by  $\sigma_x^2$ , the conditional variance, which is also the

explicit multiplier of lagged innovations. However, with standard GARCH( $p, q$ ) models volatility is calculated by the summation of lagged error terms  $\varepsilon_x$ . EGARCH( $p, q$ ) model volatility responds differently to good and bad news. The same applies to the intensity and covariance-stationary parameters, as well as  $Z_x$ .

In this study, the Maximum Likelihood method is used to estimate GARCH( $p, q$ ) parameters.

### 2.3. Heavy-tailed distributions

This section describes the Generalized Pareto Distribution and Nolan's  $S_0$ -parameterization univariate stable distribution.

#### 2.3.1. The Generalized Pareto Distribution (GPD)

The two-parameter Generalized Pareto Distribution (GPD), with the shape parameter  $\xi$  and scale parameter  $\beta$ , is interpreted by Tsay (2013), proposed from Pickands (1975), as:

$$G_{\xi\beta}(y) = \begin{cases} 1 - \left(1 + \frac{\xi y}{\beta}\right)^{-1/\xi} & \text{if } \xi \neq 0 \\ 1 - \exp\left(-\frac{y}{\beta}\right) & \text{if } \xi = 0 \end{cases}, \quad (4)$$

where  $y > 0$  when  $\xi \geq 0$ ,  $0 \leq y \leq -\beta / \xi$ , when  $\xi < 0$  and  $\beta > 0$ .

#### 2.3.2. Excess distribution

Excess function  $F_u$  for random variable  $Y$ , above a threshold  $u$ , is defined as:

$$F_u(y) = P(Y - u \leq y | Y > u),$$

where the size of exceedances over  $u$  is represented by  $y$ . Additionally, if  $F$  is denoted as distribution of  $Y$ , then:

$$F_u(y) = \frac{F(y + u) - F(u)}{1 - F(u)}. \quad (5)$$

Balkema and De Haan (1974) and Pickands (1975), identified by a fundamental theorem in Extreme Value Theory, detect asymptotic properties of exceedances with GPD.

The peaks-over-threshold (POT) method is used to estimate the threshold of the GPD (Embrechts et al., 1997). In this paper, the maximum likelihood estimation method is used to determine GPD parameters.

### 2.4. Univariate stable distributions

Stable laws were introduced by the groundbreaking work of Levy (1925), and stable distributions are a four-parameter family of models that generalize the normal distribution based on stable laws. Stable distributions are described by parameters, namely  $\alpha$ ,  $\beta$ ,  $\gamma$ , and  $\sigma$ .

Various parameterizations are used to define stable laws. The notation  $S(\alpha, \beta, \gamma, \sigma; k)$  is used to describe the class of stable laws. The  $\alpha$ ,  $\beta$ ,  $\gamma$ , and  $\sigma$  parameters are unknown and should be estimated. The integer  $k$  distinguishes between the different parameterizations (Nolan, 2013).

Nolan (2013) describes the  $S_0$ -parameterization as follows:

A random variable  $Y$  is  $S(\alpha, \beta, \gamma, \sigma; 0)$  if

$$Y \stackrel{d}{=} \begin{cases} \gamma \left( Z - \beta \tan \frac{\pi\alpha}{2} \right) + \delta, & \alpha \neq 1 \\ \gamma Z + \delta, & \alpha = 1 \end{cases}. \quad (7)$$

where  $Z \equiv Z(\alpha, \beta)$ .

Nolan (2013) recommends using the  $S_0$ -parameterization for statistical inferences and numerical purposes, since it has the simplest form for the characteristic function that is continuous in all four parameters. Nolan (2013) suggested that the maximum likelihood method is the most frequently used one to estimate stable parameters. In this study, stable parameters are estimated using the maximum likelihood method, and thereafter the Anderson-Darling goodness-of-fit test is applied to test the suitability of the fitted univariate stable models.

### 2.5. Value-at-Risk

Capital requirements of financial institutions are based on VaR estimates. VaR is a risk manage-

ment tool that has become a benchmark for measuring market risks. The main goal of estimating VaR is to assess the largest possible loss for a portfolio over a certain period of time. The quantile of a distribution is used to estimate the VaR values. For a random variable  $Y$ , the VaR estimate at given probability  $p$  is defined as the  $p$ -th quantile of  $F$ , that is:

$$VaR_p = F^{-1}(1-p), \quad (8)$$

where  $F^{-1}$  is the quantile function.

To examine the effectiveness and adequacy of VaR, various backtesting procedures are utilized. Formal conclusions on model robustness can be obtained using the Kupiec likelihood ratio (Kupiec, 1995).

In this study, the Kupiec likelihood ratio test evaluates model robustness. According to the Kupiec likelihood ratio test, a good model should have its proportion of violations of VaR estimates close to the corresponding tail probability  $\alpha$  (Kupiec, 1995).

The next section dives into the data exploration and empirical analyses.

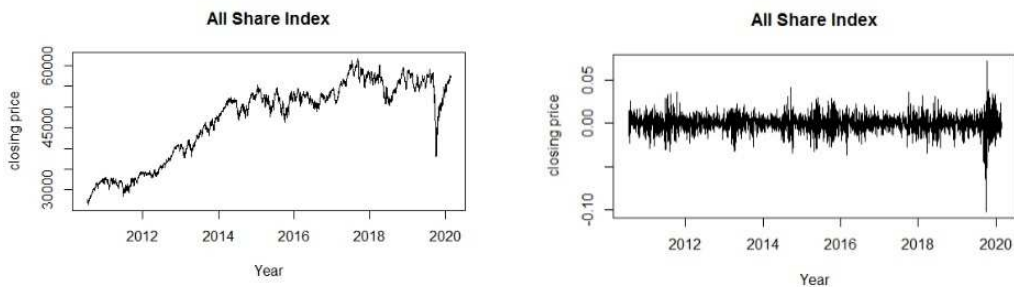
### 3. DATA EXPLORATION

In this section, the characteristics of the FTSE/JSE financial stock indices and the USD/ZAR exchange rate are described, thereafter, robust VaR models are selected for each return series. This study investigates the performance of GARCH models with Generalized Pareto distribution (GPD) and stable distributions (SD) with the estimation and forecasting of VaR within the South African financial industry.

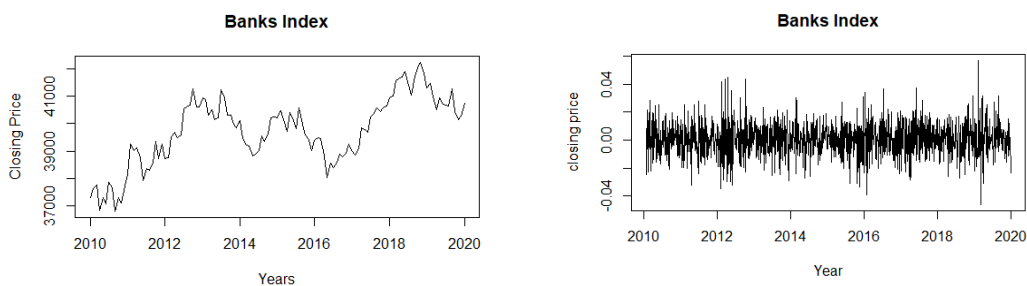
This study has data sets that consist of the daily closing prices of FTSE/JSE ALSI, FTSE/JSE Banks Index, FTSE/JSE Mining Index and USD/ZAR exchange rate prices obtained from McGregor BFA and recorded over the period from August 13, 2010 to August 14, 2020. The return series for each index is calculated as the first backward differences of the index values' natural logarithm. For day  $t$ , the daily log return  $r_t$  is defined as

$$r_t = \ln(P_t) - \ln(P_{t-1}), \quad (10)$$

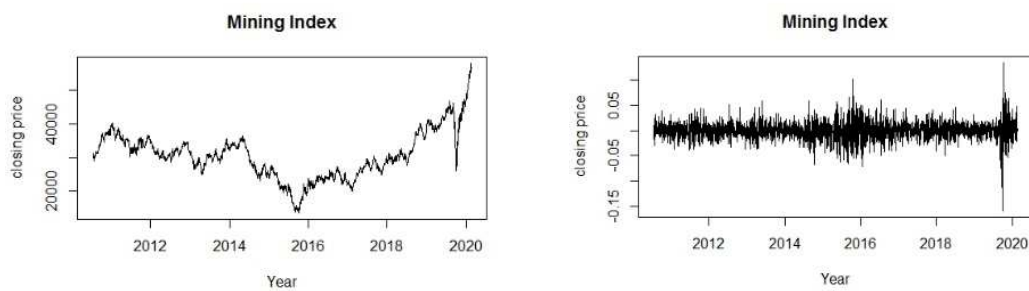
where  $P_t$  is the price at day  $t$ . Time series plots for both daily closing prices and returns are shown in Figures 1 to 4.



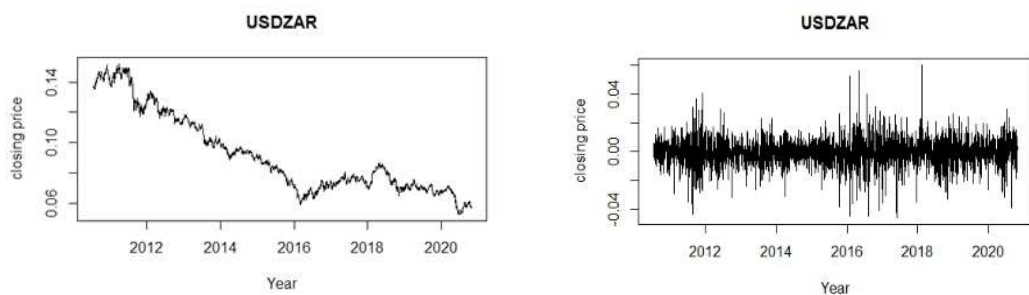
**Figure 1.** Time series plot of FTSE/JSE All Share Index prices (left) and one-day returns (right)



**Figure 2.** Time series plot of FTSE/JSE Banks Index prices (left) and one-day returns (right)



**Figure 3.** Time series plot of FTSE/JSE Mining Index prices (left) and one-day returns (right)



**Figure 4.** Time series plot of USD/ZAR exchange rate prices (left) and one-day returns (right)

In Figures 1-4, the plots seem to show various trends in mean and variance over time, indicating non-stationarity. The log returns are stationary as the mean hovers around 0, however, the variance varies over time, indicating heteroscedasticity and volatility clustering, which is likely when dealing with financial data. Isolated extreme returns

caused by shocks to financial markets are visible, such as the 2015 stock market crash and the 2019–2020 global COVID-19 pandemic.

Descriptive statistics for the daily closing prices of the financial stock indices returns and USD/ZAR exchange rate returns are shown in Panel A of

**Table 1.** Descriptive statistics of financial stock market indices and exchange rate price returns

Panel A								
Basic statistics	FTSE/JSE ALSI	FTSE/JSE Banks Index	FTSE/JSE Mining Index	USD/ZAR exchange rate				
No. of observations	2,499.00	2,499.00	2,499.00	2,675.00				
Minimum	-0.1023	-2.3021	-0.1589	-0.0460				
Maximum	0.0726	0.0991	0.1346	0.0603				
Mean	0.0003	-0.0008	0.0002	-0.0003				
Median	0.0006	0.0005	0.0003	0.0000				
Skewness	-0.7310	-41.0875	-0.1605	-0.1671				
Excess kurtosis	8.8822	1919.8771	6.0443	2.7766				
Panel B								
Testing for normality, autocorrelation and heteroscedasticity								
Test	Statistic	p-value	Statistic	p-value	Statistic	p-value	Statistic	p-value
Jarque-Bera	8,455.2667	< 0.0001	385,117.132	< 0.0001	3,823.9300	< 0.0001	874.3992	< 0.0001
Ljung Box	67.2900	< 0.0001	7.0024	0.9967	47.6200	0.0005	11.7751	0.9236
ARCH LM Test	936.0966	< 0.0001	0.0000	0.9900	50.0300	0.00	15.4100	0.0000
Panel C								
Testing for unit root and stationarity								
Unit root test	Statistic	p-value	Statistic	p-value	Statistic	p-value	Statistic	p-value
ADF Test	-13.6259	0.01	-12.9576	0.01	-13.5509	0.01	-14.7402	< 0.0001
PP Test	-2,586.908	0.01	-2,548.2810	0.01	-2,430.247	0.01	-2,688.0280	< 0.0001
KPSS Test	0.1303	0.10	0.0914	0.10	0.2394	0.10	0.0431	0.1000

Table 1. Positive averages for FTSE/JSE ALSI and FTSE/JSE Mining Index returns indicate a slight increasing trend over time, whereas the negative averages for the FTSE/JSE Banks Index and USD/ZAR exchange rate returns indicate a slight decreasing trend over time for the return series.

Tests for normality, autocorrelation and heteroscedasticity are shown in Panel B. The null hypothesis of normality for the Jarque-Bera test is rejected at the 5% level of significance for all returns. This implies that the use of heavy-tailed models should be considered when analyzing the returns series.

The significant  $p$ -values of the Ljung-Box test for FTSE/JSE ALSI and the FTSE/JSE Mining Index returns suggest the failure to reject the null hypothesis of no autocorrelation. The fitting of a statistical distribution usually assumes homoscedasticity and no autocorrelation.

Results for the unit root and stationary tests are shown in Panel C. At a 5% level of significance, the null hypothesis of a unit root is rejected, and it can be concluded that all return series are stationary. The KPSS test showed that all returns are stationary, since all  $p$ -values are 0.1, which is greater than 0.05, therefore null hypothesis of stationarity is accepted.

## 4. RESULTS

In this study, the best GARCH( $p,q$ )-type model that adequately captures volatility clustering in the return series is investigated. First, the GARCH (1,1) model with normal innovations using the Maximum Likelihood (ML) method to the return series is fitted. The ML parameter estimates and their corresponding  $p$ -values of the  $t$ -statistics for the GARCH (1,1) model are shown in Table 2 Panel A. In Panel B of Table 2, the  $p$ -values of Ljung-Box and the ARCH-LM test statistics are reported.

From Table 2, the coefficient of  $\hat{\beta}_1$  for the stock returns under investigation is significant ( $p$ -value  $> 0.05$ ). This relates to the measurement of volatility persistence, which indicates that the period of high volatility is followed by periods of low volatility, and the converse is also true. The  $p$ -values of the Ljung-Box are greater than 0.05, therefore, at a 5% level of significance, there is evidence to reject the null hypothesis of serial correlation as the GARCH (1,1) model with normal innovations indicates that each financial stock returns are not serially correlated. The ARCH LM test confirms ARCH effects of each return series.

The sign bias test is implemented in Table 3 to test if returns can be modelled by asymmetric distribution.

**Table 2.** ML parameter estimates for GARCH (1,1) model with normal innovations and corresponding goodness-of-fit statistics for Financial stock returns

Parameter estimates	FTSE/JSE ALSI	FTSE/JSE Banks Index	FTSE/JSE Mining Index	USD/ZAR exchange rate
<b>Panel A</b>				
<b>Financial stock returns</b>				
$\hat{\alpha}_0$	0.0000	0.0000	0.0000	0.0000
$\hat{\alpha}_1$	0.0884	0.1479	0.0583	0.0442
$\hat{\beta}_1$	0.8891	0.7691	0.9289	0.9447
<b>Panel B</b>				
<b><math>p</math>-values of the corresponding to the parameter estimates or test statistics</b>				
$\hat{\alpha}_0$	0.0466	0.0000	0.0549	0.1417
$\hat{\alpha}_1$	0.0000	0.0000	0.0000	0.0000
$\hat{\beta}_1$	0.0000	0.0000	0.0000	0.0000
Ljung-Box	0.1308	0.8106	0.1455	0.2262
ARCH-LM	0.5060	0.9833	0.1892	0.1726



**Table 3.** Sign bias test of return series

Sign-bias estimates	FTSE/JSE ALSI	FTSE/JSE Banks Index	FTSE/JSE Mining Index	USD/ZAR exchange rate
<b>Panel A</b>				
<b>Financial stock returns</b>				
Sign-bias	1.5437	1.1174	0.7800	0.5062
Negative sign-bias	0.1009	0.0020	1.3182	0.6277
Positive sign-bias	2.5654	0.5379	1.1302	1.6774
Joint effect	24.3508	1.2914	4.0531	4.1492
<b>Panel B</b>				
<b>p-values of the corresponding to Sign-bias estimates</b>				
Sign-bias	< 0.0001	0.2639	0.4355	0.6127
Negative sign-bias	< 0.0001	0.9984	0.1876	0.5303
Positive sign-bias	< 0.0001	0.5907	0.2583	0.0936
Joint effect	< 0.0001	0.7312	0.2558	0.2458

The sign bias test shows  $p$ -values greater than 0.05, except for the FTSE/JSE ALSI. The negative and positive sign bias  $p$ -values greater than 0.05 imply that there were significant positive and negative reaction shocks to the returns, with an exception of FTSE/JSE ALSI. The joint effect was significant for ALSI, implying that returns are asymmetrical. Table 3 shows that ALSI returns are asymmetric and best captured by asymmetric GARCH-type models.

Table 4 shows the results of fitting an EGARCH(1,1), APARCH(1,1), and TGARCH(1,1), models to the FTSE/JSE ALSI returns.

Table 4 presents the results of fitting the EGARCH(1,1), APRACH(1,1) and TGARCH(1,1) models to the FTSE/JSE ALSI returns. Based on AIC and BIC values, a GARCH-type model is the best asymmetric model for a set of returns, while the EGARCH(1,1) model is for the FTSE/JSE ALSI returns.

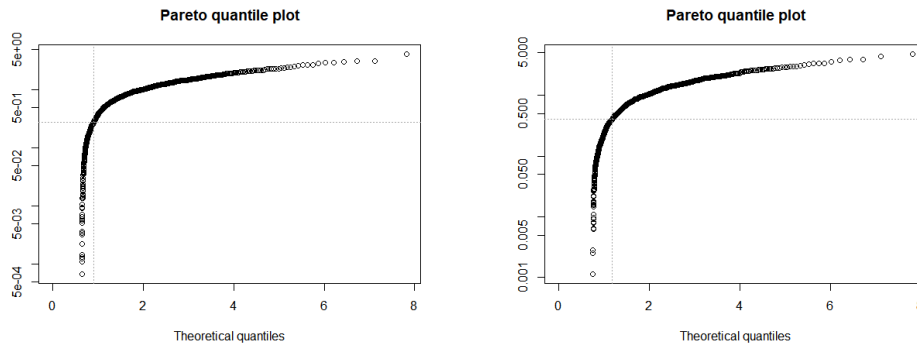
**Table 4.** ML parameter estimates for asymmetric GARCH-type models with normal innovations on ALSI returns

Parameter estimates	Asymmetric GARCH-type model		
	EGARCH(1,1)	APARCH(1,1)	TGARCH(1,1)
<b>Panel A</b>			
<b>FTSE/JSE ALSI returns</b>			
$\hat{\alpha}_0$	-0.2015	0.0002	0.0000
$\hat{\alpha}_1$	-0.1342	0.0682	0.0000
$\hat{\beta}_1$	0.9783	0.9244	0.9070
$\hat{\gamma}_1$	0.0863	1.0000	-
$\hat{\delta}_1$	-	1.0000	-
$\hat{\eta}_{11}$	-	-	0.1397
<b>Panel B</b>			
<b>p-values of the corresponding to the parameter estimates or AIC/BIC</b>			
$\hat{\alpha}_0$	0.0000	0.0000	0.0025
$\hat{\alpha}_1$	0.0000	0.0000	0.0000
$\hat{\beta}_1$	0.0000	0.0000	0.0000
$\hat{\gamma}_1$	0.0000	0.0000	-
$\hat{\delta}_1$	-	0.0000	-
$\hat{\eta}_{11}$	-	-	0.0000
AIC	-6.5637	-6.5603	-6.5594
BIC	-6.5520	-6.5486	-6.5478

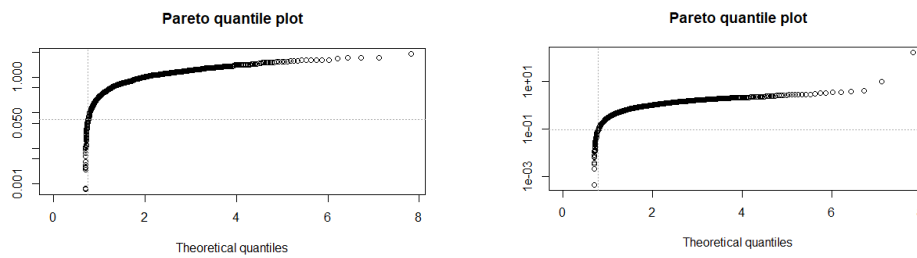
### 4.1. Fitting the hybrid-GARCH-type-GPD models

The standardized residuals from the GARCH-type model were extracted, and the GPD was fitted using the ML estimation procedure. The new model is a hybrid model of a GARCH-type

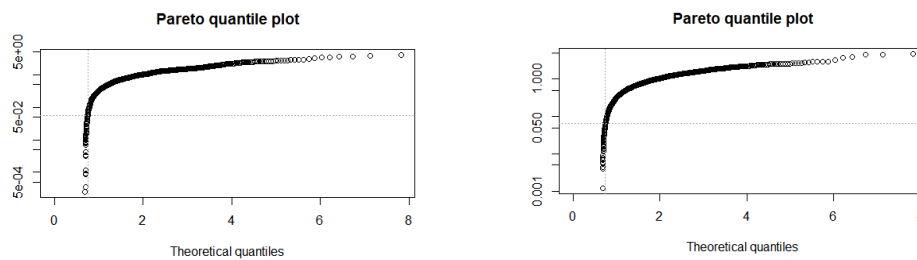
model and GPD, here referred to as a hybrid-GARCH-type-GPD model. The GPD model is fitted to the upper (gains) and lower tails (loses). To select thresholds, a mean residual life plot and parameter stability plot will be utilized. The Pareto Quantile plot will also be used to confirm the threshold.



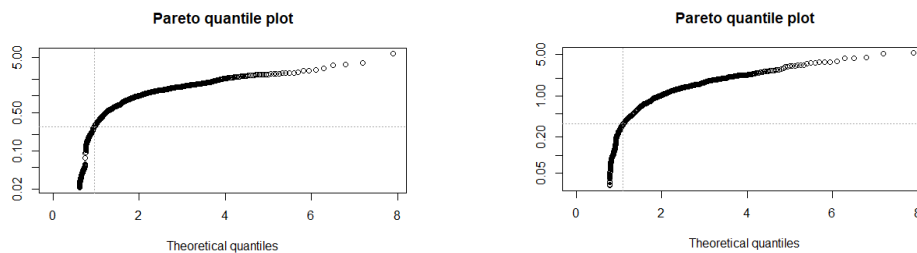
**Figure 5.** FTSE/JSE ALSI Returns Pareto quantile plot of positive (left) and negative (right) standardized residuals



**Figure 6.** FTSE/JSE Banks Index Returns Pareto quantile plot of positive (left) and negative (right) standardized residuals



**Figure 7.** FTSE/JSE Mining Index Returns Pareto quantile plot of positive (left) and negative (right) standardized residuals



**Figure 8.** USD/ZAR Exchange Rate Returns Pareto quantile plot of positive (left) and negative (right) standardized residuals

**Table 5.** ML parameter estimates of hybrid GARCH-type-GPD model

Returns	FTSE/JSE ALSI		FTSE/JSE banks index		FTSE/JSE mining index		USD/ZAR exchange rate	
	Upper	Lower	Upper	Lower	Upper	Lower	Upper	Lower
Tail	1.32	1.5	1.06	1.09	1.06	1.06	1.31	1.4
Threshold ( $u$ )								
Number of exceedances	188	172	313	313	306	M334	200	211
$\hat{\varepsilon}$	0.4499	0.6640	0.5282	0.5019	0.5752	0.6278	0.4558	0.6064
SE( $\hat{\varepsilon}$ )	0.0456	0.0699	0.0445	0.0371	0.0469	0.0448	0.0469	0.0630
$\hat{\sigma}$	-0.0172	-0.0687	0.0007	0.2584	-0.0370	-0.0491	0.0905	0.0649
SE( $\hat{\sigma}$ )	0.0706	0.0728	0.0624	0.0497	0.0582	0.0460	0.0750	0.0779

Figure 9 and Figure 10 below show the respective GPD model diagnostic plot for the positive and negative FTSE/JSE ALSI residuals.

### 4.2. ALSI GPD model diagnostic plots

The PP and QQ plot do not seem to show any major divergences from the straight lines, and the return level and density plots show that positive and negative standardized residuals follow a GPD in both the lower and upper tails for the FTSE/JSE All Share Index. Similar deductions are made for all return series observed in this study.

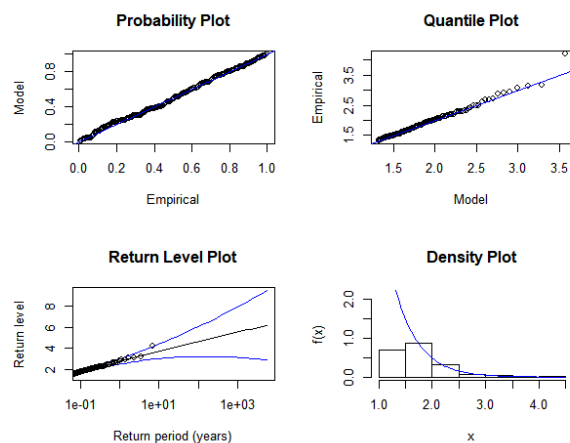
### 4.3. Fitting the hybrid GARCH-type-SD models

Standardized residuals of the best GARCH-type model are extracted, and stable parameters are estimated under Nolan's  $S_0$  - parameterization using maximum likelihood estimation resulting in a GARCH-type model combined with SD governing the innovations. This model is referred to as the hybrid GARCH-type-SD model. Table 6 reports the ML parameter estimates of SD fitted to the standardized residuals extracted from the suitable GARCH-type model.

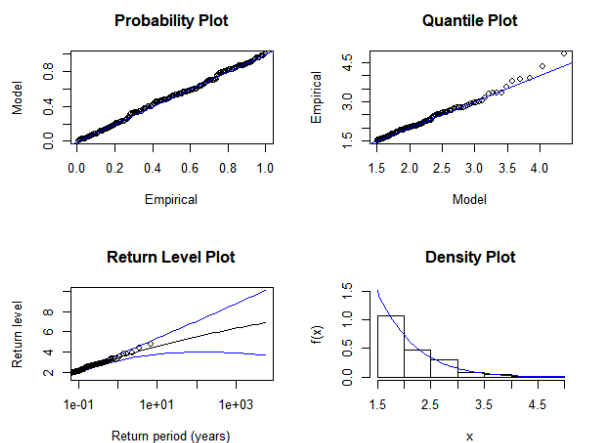
**Table 6.** ML parameter estimates of the hybrid GARCH-type-SD model

Parameter estimates	Financial stock returns			
	FTSE/JSE ALSI	FTSE/JSE Banks Index	FTSE/JSE Mining Index	USD/ZAR exchange rate
$\hat{\alpha}$	1.9012	1.9052	1.9149	1.8474
$\hat{\beta}$	-0.8719	-0.1299	-0.1567	-0.5224
$\hat{\gamma}$	0.6689	0.6586	0.6714	0.6396
$\hat{\delta}$	0.0834	-0.0102	-0.0028	0.0634
AD Test	0.8937	—*	0.5147	1.5819
(p-value)	(0.4181)	(< 0.0001)	(0.7319)	(0.1581)

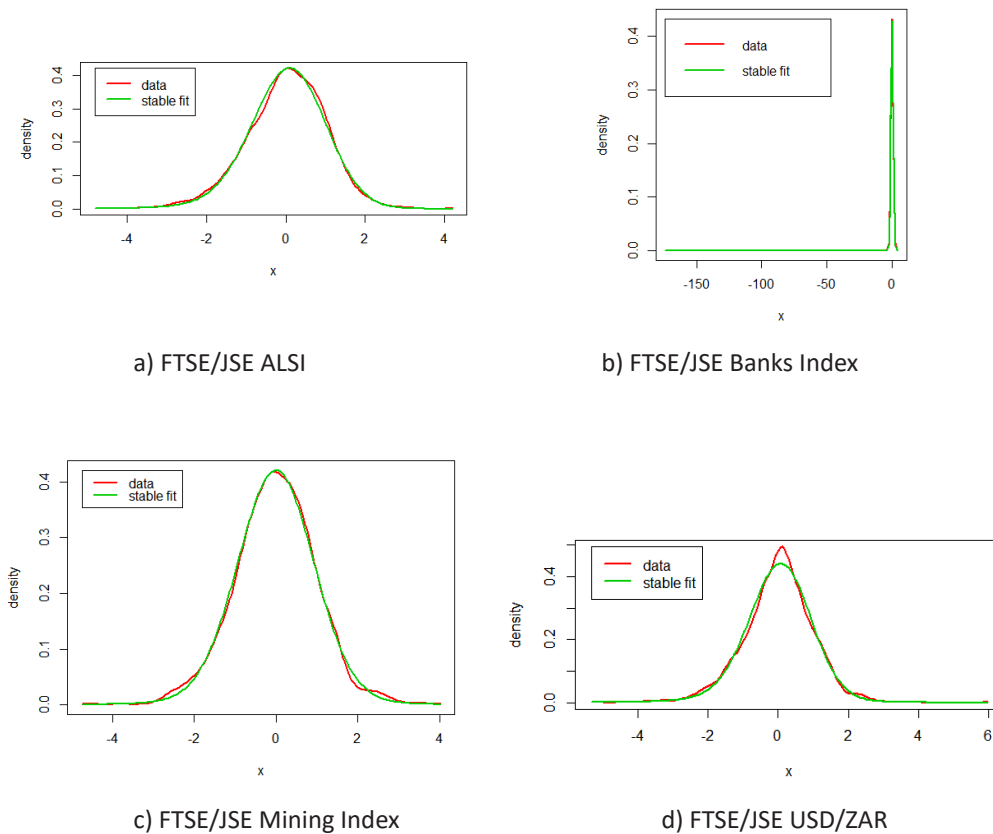
Note: \* No AD statistic available for FTSE/JSE Banks Index.



**Figure 9.** GPD positive ALSI residuals



**Figure 10.** GPD negative ALSI residuals



**Figure 11.** Stable density plots of return series

Stable density plots in Figure 11 indicate that the estimated univariate  $S_0(\alpha, \beta, \gamma, \delta)$  model is adequate in describing the residuals extracted through the GARCH-type model.

#### 4.4. VaR and backtesting

VaR values are calculated at different levels. Table 7 shows VaR estimates at different VaR levels of the suitable hybrid GARCH-GPD and GARCH-SD models of the financial market and USD/ZAR exchange rate returns.

Table 7 shows that the VaR models resulting from VaR estimates are within the same ranges for short and long positions. The VaR estimates from the fitted hybrid GARCH-type models are backtested using the Kupiec likelihood ration test. The  $p$ -values of the Kupiec likelihood ration test at different VaR levels are summarized in Table 8.

Table 8 shows the  $p$ -values of the Kupiec likelihood ration test, the fitted GARCH-GPD and GARCH-SD models are well suited to the return series observed in this study at almost all VaR levels, since

**Table 7.** VaR estimates of the financial market indices and exchange rate price returns using fitted hybrid GARCH-GPD and GARCH-SD model

Financial stock returns	Hybrid Model	VaR estimates at different levels			
		Short position		Long position	
		2.5%	5%	95%	97.5%
FTSE/JSE ALSI	EGARCH(1,1)–GPD	1.8110	1.5032	1.7099	2.1495
	EGARCH(1,1)–SD	–2.0928	–1.6655	1.5604	1.8478
FTSE/JSE Banks Index	GARCH(1,1)–GPD	1.9116	1.5451	1.61010	2.0931
	GARCH(1,1)–SD	–1.9652	–1.6100	1.5513	1.8868
FTSE/JSE Mining Index	GARCH(1,1)–GPD	1.9476	1.5668	1.6627	2.0703
	GARCH(1,1)–SD	–1.9859	–1.6295	1.5818	1.9170
USD/ZAR exchange rate	GARCH(1,1)–GPD	1.8350	1.4968	1.6806	2.1234
	GARCH(1,1)–SD	–2.0848	–1.6355	1.5124	1.8219

**Table 8.**  $p$ -values of the Kupiec likelihood ratio test for financial indices and exchange rate returns

Financial stock returns	Hybrid Model	$p$ -values of the Kupiec likelihood ratio test			
		Short position		Long position	
		2.5%	5%	95%	97.5%
FTSE/JSE ALSI	EGARCH(1,1)-GPD	0.7479	0.6851	0.9304	0.5618
	EGARCH(1,1)-SD	0.9464	0.5212	0.0480	0.4766
FTSE/JSE Banks Index	GARCH(1,1)-GPD	0.9514	0.5527	0.3170	0.3985
	GARCH(1,1)-SD	0.2849	0.3170	0.7156	0.5665
FTSE/JSE Mining Index	GARCH(1,1)-GPD	0.7496	0.3288	0.9963	0.5665
	GARCH(1,1)-SD	0.2849	0.5212	0.0600	0.9514
USD/ZAR exchange rate	GARCH(1,1)-GPD	0.8156	0.5356	0.9823	0.8895
	GARCH(1,1)-SD	0.4547	0.4685	0.6717	0.8156

the observed  $p$ -values are greater than 0.05. Hence, the null hypothesis of model adequacy is accepted. The GARCH-GPD model suggests model inadequacy for the 99% VaR level for the Banks Index returns, whereas the fitted GARCH-SD models show model inadequacy for the Mining Index returns at the 99% VaR, when considering a 5% level of significance. However, with a 1% level of significance, the robustness of the fitted GARCH-SD models is highlighted at all VaR levels, whereas the fitted GARCH-GPD still displays model inadequacy at the 99% VaR level. The most robust VaR model is summarized at each VaR level. Table 9 shows the most appropriate hybrid GARCH-type model selected at different VaR levels for the returns of the financial indices and USD/ZAR exchange rate returns.

It is clear from Table 9 that there is no hybrid model that outperforms another at all VaR levels for all financial stock returns. For FTSE/JSE ALSI, the EGARCH(1,1)-GPD model is the most robust VaR model at the 5% short position and at all long positions. For FTSE/JSE Banks index, GARCH(1,1)-SD is the robust model at all levels of the long position, and the GARCH(1,1)-GPD model is the robust model at all levels of the short position. For FTSE/JSE Mining index, the GARCH(1,1)-GPD is

the best VaR model at the 2.5% VaR level at short position and 5% VaR level at long position. At the 5% VaR level short position and 2.5% VaR level at long position, the GARCH(1,1)-SD is the most appropriate VaR model for FTSE/JSE Mining index. The GARCH(1,1)-GPD model performs best at all VaR levels for USD/ZAR exchange rate returns.

## 5. DISCUSSION

Preliminary tests were conducted to investigate the nature of each return series and to highlight the stylized properties of financial data such as volatility clustering and heteroscedasticity. In this study, each FTSE/JSE indices and USD/ZAR exchange rate showed leptokurtic behavior, stationarity and rejected the hypothesis of normality. Serial correlation is evident in the FTSE/JSE Banks Index and USD/ZAR exchange rate returns, which opposes the assumption of no serial correlation and heteroscedasticity when fitting a suitable statistical distribution. However, McNeil, et al (2015) showed that some financial return series exhibit serial correlation. Evidence for empirical properties of financial data with ARCH effects in reach return series implies the use of the GARCH-type framework

**Table 9.** Most appropriate hybrid GARCH-type model selected for financial indices and USD/ZAR exchange rate returns at different VaR levels

Financial stock returns	Selected hybrid GARCH-type models at different VaR levels			
	Short position		Long position	
	2.5%	5%	95%	97.5%
FTSE/JSE ALSI	EGARCH(1,1)-SD	EGARCH(1,1)-GPD	EGARCH(1,1)-GPD	EGARCH(1,1)-GPD
FTSE/JSE Banks	GARCH(1,1)-GPD	GARCH(1,1)-GPD	GARCH(1,1)-SD	GARCH(1,1)-SD
FTSE/JSE Mining	GARCH(1,1)-GPD	GARCH(1,1)-SD	GARCH(1,1)-GPD	GARCH(1,1)-SD
USD/ZAR Exchange rate	GARCH(1,1)-GPD	GARCH(1,1)-GPD	GARCH(1,1)-GPD	GARCH(1,1)-GPD

with heavy-tailed distributions Generalized Pareto Distribution (GPD) and Nolan's  $S_0$ -parameterization stable distribution (SD).

The suitability of the GARCH(1,1) model for each return series is confirmed, except for FTSE/JSE ALSI where asymmetric effects are present. EGARCH(1,1) was found to best capture volatility present in FTSE/JSE ALSI returns.

The GPD model fitted to the tail distribution has been shown to capture asymmetry in tails in line with McNeil and Frey (2000), which uses a conditional method for estimating tail innovations of a GARCH-type model using a heavy-

tailed distributions recommended by extreme value theory (EVT).

The stable density plots and model diagnostics suggest a good fit of the residuals for each financial return series similar to Nolan (2003), thus suggesting a strong alternative to modelling the tail behavior of fitted GARCH-type models. The fitted GARCH-type SD model outperforms the fitted GARCH-type GPD model at various VaR levels using the Kupiec likelihood backtesting procedure, especially in the upper tail (gains) for the FTSE/JSE indices. Therefore, the robustness of a fitted hybrid GARCH-type SD model is highlighted when considering tail innovations of South African financial data.

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## CONCLUSION

This paper analyzes daily returns of four South African data sets, namely, FTSE/JSE ALSI, FTSE/JSE Banks index, FTSE/JSE Mining index and the USD/ZAR exchange rate. The GARCH(1,1) model was fitted to each return series, and the sign bias showed asymmetry for FTSE/JSE ALSI only. Thereafter, the GPD and Nolan's  $S_0$ -parameterization stable models were fitted to the best GARCH-type model residuals. Model diagnostics show that both the GARCH(1,1)-GPD and GARCH(1,1)-SD models provide a parsimonious fit to the South African financial data.

VaR estimates and VaR in-sample backtesting using the Kupiec likelihood ratio test emphasize the robustness of the fitted GARCH(1,1)-SD model in both long and short positions. Stable distributions are largely justified in finance, since they capture large fluctuations, as well as various other empirical properties, for example, heavy tails, asymmetry and volatility that are prevalent in financial data. Parties concerned with extreme losses such as risk analysts, insurers, policy makers, and risk-averse investors may benefit from the use of GARCH(1,1)-GPD and GARCH(1,1)-SD models as an alternative to modelling financial series. As further research, possible alternative criteria for evaluating and selecting a data tail-based model, such as the hybrid GARCH-type mixture model of GPD-Normal-SD, where GPD estimates VaR values in a short position and SD estimates VaR values in a long position jointly, are recommended.

## AUTHOR CONTRIBUTIONS

Conceptualization: Kimera Naradh.

Formal analysis: Kimera Naradh, Knowledge Chinhamu, Retius Chifurira.

Investigation: Kimera Naradh.

Methodology: Kimera Naradh.

Software: Knowledge Chinhamu.

Supervision: Knowledge Chinhamu, Retius Chifurira.

Validation: Kimera Naradh, Retius Chifurira.

Writing – original draft: Kimera Naradh.

Writing – review & editing: Knowledge Chinhamu, Retius Chifurira.

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