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A note on optimal insurance under ambiguity

Abstract

This paper considers the effect of ambiguity on the market for insurance when preferences are ordered by comparison of Choquet expected utilities. The author shows that Arrow's theorem on optimal insurance easily extends to the ambiguity case when the Choquet integrals are with respect to a common capacity. The paper also presents an extension of Jensen's inequality to integrals with respect to normalized capacities which can be used, among other things, in a discussion of feasibility.

Keywords: optimal insurance, ambiguity, Choquet expected utilities, Jensen's inequality.

Introduction

A decision maker faces ambiguity when he/she cannot fully determine the probability distribution that is relevant to his/her decisions. Frank Knight (1921) made a distinction between risk, where the decision maker knows or can assign probabilities to all events, and uncertainty, where the decision maker cannot assign probabilities to all events of interest. Keynes (1921, 1936) also made a distinction between risk and uncertainty and considered "animal spirits" as an important factor in the market because of lack of information to assign precise probabilities.

Most of the theoretical work in the modern financial economics literature, however, is based on the assumption of existence of probabilities (objective or subjective) that are known. The strong assumption about the knowledge of probabilities is to some extent justified by the axiomatic treatment of subjective probability by Savage (1954).

The axiomatic approach of Savage (1954) asserts that when the preferences of a decision maker who is facing uncertainty satisfy a set of rationality axioms, there exists an underlying (subjective) probability distribution guiding his/her preferences. A theorem of Savage (1954) asserts that a preference relation \leq , with the meaning of "not preferred to", is a qualitative probability under the following five postulates:

P1: The relation \leq , which means "not preferred to" is a simple ordering. Savage calls a relation R on a set X a simple ordering if for all $x \in X$ and $y \in X$, either xRy or yRx , and the relation is transitive.

P2: If f and g are two acts with payoffs α and β on an event B and equal payoff γ_1 on B^C , then changing γ_1 to γ_2 will not alter the preference ordering between f and g .

The postulate P2 is referred to as the sure thing principle.

P3: If $f(s) = g$ and $f'(s) = g'$ for all s , where $f(s)$ and $f'(s)$ are consequences corresponding to acts f and f' ,

and B is not a null set. Then $f \leq f'$ given B if and only if $g \leq g'$.

P4: If f_A, f_B, g_A , and g_B are acts such that f_A has payoff f on the event A and f' on A^C with $f' < f$; f_B has payoff f on the event B and f'' on B^C ; g_A has payoff g on the event A and g' on A^C with $g' < g$; g_B has payoff g on the event B and g'' on B^C ; $f_A \leq f_B$ then $g_A \leq g_B$.

P5: There exists a pair of consequences f and f' such that $f' < f$.

With an additional set of axioms Savage shows that there exists a unique finitely additive probability measure that strictly agrees with the qualitative probability and there exists a utility function such that comparison of the expected utility of the corresponding consequences, with the expectation based on the personal quantitative probability, gives the preference ordering of the individual.

Ellsberg (1961) performed some experiments involving prominent, sophisticated, and rational people and showed that some of his subjects, including Savage himself, had preferences that would not imply any underlying probabilities.

In one of his experiments Ellsberg posed the following question:

Suppose Urn I contains 100 black balls and red balls. We don't know how many are black, and how many are red. The number of black balls can be 0, ..., 100. Urn II has 50 red and 50 black balls. You can draw one ball. If the ball is red you will receive \$100, and if it is black you will get \$0. Do you prefer to draw the ball from Urn I or Urn II? The majority response was that they would prefer to draw the ball from Urn II.

According to Savage the probability that these subjects assign to drawing a red ball out of Urn I is less than the probability of choosing a red ball out of Urn II, which is 1/2.

When the question is repeated with red and black exchanging places, because of symmetry the same individuals will again prefer to draw the ball from Urn II, which according to Savage means the proba-

bility assigned to drawing a black ball out of Urn I is also less than the probability of a black ball out of Urn II which is $1/2$.

Ellsberg also performed an experiment based on drawing a ball out of an Urn containing 30 red balls and 60 black and yellow balls (with unknown proportions). He provided four options with different payoffs for the color of the ball that comes out of the Urn and asked his subjects to order selected pairs of options according to their preferences. In this experiment the majority of the subjects were violating Savage's P2.

In his paper Ellsberg states that the subjects, who expressed a strict preference of an option to another in his experiment, were not minimaxing, they were not applying a Hurwicz criterion [i.e., they were not deciding a preference ordering by looking at a weighted average of the maximum payoff and the minimum payoff for each option (see Hurwicz, 1951)] because the maximum payoffs and the minimum payoffs for the four options were equal, and they were not minimaxing regret.

One interpretation of Ellsberg's results is that the majority of individuals are ambiguity averse. Hence they would demand a premium for choosing an act that involves ambiguity.

As further empirical evidence that ambiguity actually exists in the market, Epstein and Wang (1994) mention the papers by Cragg and Malkiel (1982), Zarnowitz (1984), Ito (1990), Frankel and Froot (1990) that reject the rational expectations hypothesis. Chen and Epstein (2002) refer to the "equity premium puzzle" of Mehra and Prescott (1985) which is due to "the failure of the representative agent model to fit historical averages of equity premiums and the risk free rate". They also refer to "the home-bias puzzle" created by the reluctance of investors in many countries to invest in foreign securities.

A notable approach to generalizing the method of using expected utility for preference ordering to the ambiguity case, which has been considered in the literature, is to drop the requirement of finite additivity that a probability measure must satisfy and work with integrals of the utility function with respect to a capacity instead of integrals with respect to a probability measure.

In this paper we look at the economics of insurance when preferences are given by integrals with respect to a capacity. Our main results are an extension of Arrow's (1963) theorem on optimal insurance to the ambiguity case and an extension of the Jensen's inequality which is used in the discussion of feasibility. Section 1 provides some prelimi-

naries about capacities and some of the properties of integrals with respect to capacities which will be used in the proof of our main results. Our main results are presented in Section 2. The final section concludes the paper.

1. Preliminaries

Let Ω be a Hausdorff space. An extended real valued function ν on 2^Ω is called a regular capacity (see Choquet, 1969) if:

1. $(A \subset B) \Rightarrow (\nu(A) \leq \nu(B))$.
2. $(A_n \uparrow A) \Rightarrow (\nu(A_n) \uparrow \nu(A))$, as $n \rightarrow \infty$.
3. $(K_n \text{ is a decreasing sequence of compact sets converging to } K) \Rightarrow (\nu(K_n) \downarrow \nu(K))$, as $n \rightarrow \infty$.

The capacity ν is called a normalized capacity if $\nu(\emptyset) = 0$, and $\nu(\Omega) = 1$.

A capacity ν is called alternating of order n (see Choquet (1953-54), or Shafer (1979)) if for every A, A_1, \dots, A_n in the domain of ν such that $A \subset A_i$ for every i , with $|I|$ denoting the number of elements in the set I ,

$$\nu(A) \leq \sum \{(-1)^{|I|+1} \nu(\bigcup_{i \in I} A_i) \mid \emptyset \neq I \subset \{1, \dots, n\}\}.$$

A capacity that is alternating of order 2 is also called 2-alternating. It follows from the definition that for a 2-alternating capacity $\nu(A \cup B) \leq \nu(A) + \nu(B) - \nu(A \cap B)$ for all A and B in the domain of ν .

A capacity ν is called monotone of order n , if for every A, A_1, \dots, A_n in the domain of ν such that $A_i \subset A$,

$$\nu(A) \geq \sum \{(-1)^{|I|+1} \nu(\bigcap_{i \in I} A_i) \mid \emptyset \neq I \subset \{1, \dots, n\}\}.$$

A capacity ν which is monotone of order 2 is also called a 2-monotone capacity. From the definition it follows that for a 2-monotone capacity $\nu(A \cup B) \geq \nu(A) + \nu(B) - \nu(A \cap B)$ for all A and B .

Let f be a real valued function on Ω . The Choquet integral of f with respect to a normalized capacity ν (see for example Schmeidler (1986) or Denneberg (1994)) is defined by

$$\int_{\Omega} f d\nu = \int_{-\infty}^0 (\nu[f \geq t] - 1) dt + \int_0^{\infty} \nu[f \geq t] dt.$$

The Choquet integral is monotone;

$$f \leq g \Rightarrow \int_{\Omega} f d\nu \leq \int_{\Omega} g d\nu.$$

It is positive homogeneous; for $\lambda \geq 0$

$$\lambda \int_{\Omega} f d\nu = \int_{\Omega} (\lambda f) d\nu.$$

The Choquet integral is subadditive;

$$\int_{\Omega} (f + g)dv \int \leq \int_{\Omega} fdv + \int_{\Omega} gdv$$

if and only if (see Denneberg (1994), Chapter 6) it is with respect to a 2-alternating capacity.

It is superadditive;

$$\int_{\Omega} (f + g)dv \int \geq \int_{\Omega} fdv + \int_{\Omega} gdv$$

if and only if it is with respect to a 2-monotone capacity.

Two functions f and g are said to be comonotone if for every s and t

$$[f(s) - f(t)][g(s) - g(t)] \geq 0.$$

For two comonotone functions f and g (see Denneberg (1994), Page 65)

$$\int_{\Omega} (f + g)dv = \int_{\Omega} fdv + \int_{\Omega} gdv$$

The Choquet integral has been used in the study of robustness in statistics (Huber and Strassen, 1973). It is also used by Gilboa (1987) and Schmeidler (1989) to model preferences under ambiguity.

Gilboa (1987) and Schmeidler (1989) considered weakening the sure thing principle axiom of Savage (1953). They wrote different sets of axioms and arrived at preference orderings that can be represented by Choquet integrals with respect to some capacity.

Gilboa and Schmeidler (1989) suggest ordering preferences by a maximin principle that says: For each act f compute

$$J(f) = \min\{E_P u(f) : P \in C\},$$

where u is a von-Neumann-Morgenstern utility function, and C is a set of probability measures with certain properties. Then, $f \leq g$ if and only if $J(f) \leq J(g)$.

They refer to a result of Schmeidler (1986) which states if ν is a 2-monotone capacity and C is the core of ν then for real valued function f ,

$$\int fdv = \min\{E_P f : P \in C\}.$$

The core of ν is the set of finitely additive probability measures that majorize ν pointwise.

2. Feasibility and optimal insurance

One can easily check that if the maximin method is used in the computation of the net premium by an insurer to cover a random loss X the premium would equal $\max\{E_P(X) : P \in C\}$. Although the premium that is calculated in this way is a conservative premium and hence justifiable to the insurer,

it raises the question of feasibility. The following extension of Jensen's inequality provides an answer to the question of feasibility for ordering preferences by Choquet integrals as a whole and not just the maximin method.

Theorem 1 (Jensen's inequality). Let ν be a normalized capacity and let u be a non-decreasing concave function such that $u' \geq 0$, and $u'' < 0$. Let f be such that $|\int_{\Omega} fdv| < \infty$. Then,

$$\int_{\Omega} u(f)dv \leq u(\int_{\Omega} fdv). \tag{1}$$

Proof. Let $\mu = \int_{\Omega} fdv$. By concavity of u we have

$$u(f) - u(\mu) \leq u'(\mu)(f - \mu). \tag{2}$$

Consider Case 1: $\mu \geq 0$ and $u(\mu) \geq 0$. Write (2) as:

$$u(f) + u'(\mu)\mu \leq u'(\mu)f + u(\mu). \tag{3}$$

Since $u'(\mu)\mu$ and $u(\mu)$ are constants they are comonotone with every function. Taking integrals of both sides of equation (3) we have

$$\int_{\Omega} u(f)dv + \int_{\Omega} u'(\mu)\mu dv \leq \int_{\Omega} u'(\mu)fdv + \int_{\Omega} u(\mu)dv. \tag{4}$$

Since ν is a normalized capacity, equation (1) follows from equation (4) by positive homogeneous property of the integral. Similarly for Case 2: $\mu \geq 0$ and $u(\mu) < 0$ we can write equation (2) as

$$u(f) + u'(\mu)\mu - u(\mu) \leq u'(\mu)f, \tag{5}$$

and take integrals of both sides of (5) to obtain (1). For Case 3: $\mu < 0$ and $u(\mu) < 0$ we can write equation (2) as

$$u(f) - u(\mu) \leq u'(\mu)f - u'(\mu)\mu,$$

and take integrals of both sides, and finally for Case 4: $\mu < 0$ and $u(\mu) \geq 0$, we can write equation (2) as

$$u(f) \leq u'(\mu)f - u'(\mu)\mu + u(\mu),$$

and obtain equation (1) by taking integrals of both sides.

Theorem 1 gives a natural extension of a well known feasibility argument based on Jensen's inequality, see for example Bowers et al (1997), to the case where the preference orderings are given by comparison of Choquet integrals. Ignoring all frictions that exist in the market, an insurer who has wealth $\omega_I \geq 0$ and linear utility function $u_I(x) = ax + b$, with $a > 0$, will be indifferent between insuring a random loss X with a premium H and not insuring it when

$$\int_{\Omega} [a(\omega_I + H - X) + b]dv = \int_{\Omega} (a\omega_I + b)dv.$$

Hence $H = - \int_{\Omega} -Xdv$. On the other hand, a decision maker who has wealth $\omega \geq 0$ and a non-decreasing concave utility function u and faces the random loss X will be indifferent between purchasing full coverage insurance at a price G and not buying insurance if

$$\int_{\Omega} u(\omega - G)dv = \int_{\Omega} u(\omega - X)dv.$$

Note that if u is a utility function and $u^*(x) = au(x) + b$ with $a > 0$, then u and u^* give the same preference ordering when the ordering is by comparison of Choquet expectations. Therefore, without loss of generality we may assume $u(\omega - G) \geq 0$. By the positive homogeneous property of the integral and Jensen's inequality it follows that

$$\begin{aligned} u(\omega - G) &\leq u\left(\int_{\Omega} (\omega - X)dv\right) = u\left(\omega + \int_{\Omega} -Xdv\right) = \\ &= u(\omega - H). \end{aligned}$$

Hence $G \geq H$ since u is non-decreasing.

The following theorem gives an extension of Arrow's (1963) theorem on optimal insurance to the case when decisions are made under ambiguity. Let $I(x)$ be the amount that an insurance policy I pays for coverage against a loss equal to x . We identify an insurance policy I with its payoff function. In what follows we will assume that with a feasible insurance policy I , an insured cannot benefit from an increase in the amount of loss. Thus, $x - I(x)$ is non-decreasing for every feasible insurance policy I .

Theorem 2. Consider a decision maker who has wealth ω and faces a loss X and will spend an amount P on insurance. Suppose all decisions can be ordered by the Choquet expectations with respect to the same capacity, ν and the decision maker has a non-decreasing concave von Neumann-Morgenstern utility function u such that u' exists. Suppose each feasible policy $I(X)$ is a non-decreasing function of X and is sold at a net price equal to

$$P = - \int_{\Omega} -I(X)dv.$$

Then the decision maker's Choquet expected utility will be maximized by purchasing a stop-loss insurance policy.

Proof. Let $I_d(X)$ be a stop-loss insurance with deductible d , that is $I_d(X) = (X - d)1_{[X > d]}$, and let $I(X)$ be another insurance such that, with \int denoting the Choquet integral with respect to ν ,

$$- \int_{\Omega} -I_d(X) = - \int_{\Omega} -I(X) = P.$$

By the intermediate value theorem and decreasing property of u' we have, as in the proof of Arrow's theorem (c.f. Bowers et al., 1997),

$$\begin{aligned} u(\omega - X + I(X) - P) - u(\omega - X + I_d(X) - P) &\leq \\ &\leq [I(X) - I_d(X)]u'(\omega - X + I_d(X) - P) \leq \\ &\leq [I(X) - I_d(X)]u'(\omega - d - P). \end{aligned} \tag{6}$$

Rearranging equation (6) we have:

$$\begin{aligned} u(\omega - X + I(X) - P) - u'(\omega - d - P)I(X) &\leq \\ &\leq u(\omega - X + I_d(X) + P) - u'(\omega - d - P)I_d(X). \end{aligned} \tag{7}$$

Observe that $(-x + I(x))$, and $-I(x)$ are both non-increasing. Since u is non-decreasing and $u'(\omega - d - P)$ is a non-negative constant, the left hand side of equation (7) is the sum of two comonotone functions. Similarly the right hand side of equation (7) is the sum of two comonotone functions. Taking Choquet integrals of both sides of equation (7) gives the assertion of the theorem by comonotone additivity and positive homogeneous property of the Choquet integral.

Note that in Theorem 2 the assumption that $I(X)$ is a non-decreasing function of X can be replaced with the assumption that ν is a 2-monotone capacity. Under this assumption we have:

$$\begin{aligned} \int_{\Omega} u(\omega - X + I(X) - P) + \int_{\Omega} -u'(\omega - d - P)I(X) &\leq \\ &\leq \int_{\Omega} u(\omega - X + I(X) + P) - u'(\omega - d - P)I(X). \end{aligned} \tag{8}$$

The right hand side of equation (8) is less than or equal to the Choquet integral of the right hand side of equation (7). Hence, the assertion of Theorem 2 follows as before because $I_d(X)$ is a non-decreasing function of X , and therefore as explained in the proof of Theorem 2, the right hand side of equation (7) is the sum of two comonotone functions.

Conclusion

Let X_1 and X_2 represent the losses for two risks to be insured by an insurer with a linear utility function in a competitive market. Then the price of insuring X_1 and X_2 together has to be less than or equal to the sum of prices of insuring them separately because otherwise the insurer will purchase separate policies for X_1 and X_2 from different insurers. This means if an insurer with a linear utility function is using a capacity ν to determine net premiums, then

$$- \int_{\Omega} -(X_1 + X_2)dv \leq - \int_{\Omega} -X_2dv - \int_{\Omega} -X_2dv$$

for all X_1 and X_2 .

Hence ν must be a 2-monotone capacity.

Let P_0 denote the joint probability distribution of all random variables under consideration. When ambigu-

ity does not exist, the usual assumption in financial economics is that P_0 is known to all agents and hence it is known to all sellers and buyers of insurance. Consider the ambiguity case when it is only known that P_0 belongs to a given set of probability measures M and all agents have the same information. Let ν be the set function defined by:

$$\nu(A) = \inf\{P(A) : P \in M\}.$$

Then under certain conditions, see for example Kadane and Wasserman (1996), ν is a belief function which is a monotone capacity of order 2. Since all agents have the same information, ν may be considered as the common capacity that is used for ordering preferences.

The main results of this paper include an extension of Jensen's inequality to Choquet integrals with respect to normalized capacities and an extension of Arrow's (1963) theorem on optimal insurance to the

case where preferences are ordered by comparison of Choquet expected utilities. The extension of Arrow's theorem is obtained under the assumption that (1) the amount paid by a feasible insurance policy for coverage against a loss X does not decrease with an increase in the amount of the loss, and (2) a feasible insurance policy does not cause moral hazard. Part (2) of the assumption implies that a policyholder cannot benefit from an increase in the amount of the loss, hence the policyholder's share of the loss ($X - I(X)$) does not decrease when the amount of the loss increases. We have shown that part (1) of the assumption can be replaced with the assumption that the Choquet integrals, which give the preference orderings, are with respect to a 2-monotone capacity. The latter assumption is justified by the market requirement that the cost of insuring two risks together cannot be more than the cost of insuring them separately.

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