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Financial risk management and the rational lapse strategy in life insurance policies

Abstract

Over the past decade, Variable Annuities have experienced tremendous growth accounting for half of the life insurance industry, as unit-linked products offering both participation in equity market and guarantees at key life moments (retirement, death). The recent Quantitative Impact Study (QIS 5) of the Solvency II framework showed that lapse risk is the most important risk among life underwriting risks for Variable Annuities, as illustrated by solvency issues experienced by the policyholder run in the late 1980's. Thus research on lapse rates is crucial to a proper calibration of regulatory standard models and internal risk models.

Usually the lapse behavior has been modeled by historical or backward looking statistical regressions which have empirically underestimated the risk due to the scarcity of extreme scenario samples and the inability to dynamically extrapolate the observed behavior to various market conditions. In contrast, a "rational" lapse strategy valuation is a prudent forward looking approach where policyholders lapse in a way that maximizes the net present value of the future cash-flows, depending on key drivers. Empirically consistent with herd behavior as experienced in the last financial crisis, this approach is illustrated on a GMAB VA product using two alternatives numerical schemes (PDE and Monte Carlo).

However, as policyholders cannot be expected to lapse all at the same time, this rational lapse framework is slightly amended by introducing a proportion of lapses among the contract still active, which translates into the notion of "reasonable" lapse more consistent with empirics.

Keywords: GMAB, Variable Annuity, rational lapse strategy, stochastic interest, PDE, ADI, high-dimensional regression.

Introduction

The VA product, a popular retirement savings vehicle in the US, is starting to emerge as a viable option in other markets, including Europe and Asia. The GMAB riders written on VAs (also known as Maturity Guarantees, see [6]) provide policyholders a guaranteed amount at a fixed expiration date, so this kind of products have some similar properties as long-term vanilla puts. One important attractiveness of GMAB products is that this guarantee gives policyholders the ability to protect their retirement investments against downside market risk by allowing the policyholder to receive the greater of the account value and the benefit base at the maturity. The benefit base can either step up to the high-water mark of the account value at the end of each policy year (annual ratchet), or can roll up with a fixed percentage (the roll-up rate, e.g. 2%), regardless of the market conditions. Thanks to these new product characteristics, the guarantee not only protects policyholders against investment losses, but also allows customers taking advantage of upside gain from the market. In exchange for this benefit, the policyholder pays a charge fee each year.

The recent Quantitative Impact Study (QIS 5) of the Solvency II framework showed that lapse risk is the most important risk among life underwriting risks for Variable Annuities, as illustrated by solvency issues experienced by the policyholder run in the late 1980's. Thus research on lapse rates is crucial to a proper calibration of regulatory standard models and internal risk models.

The dynamic behavior is essentially a selection process of the policyholders against the VA writer, where an increase leaves fewer insured to ultimately make a claim on the guarantees but reduces the fees the insurer can collect. The general pattern is that more policies will lapse when the capital market is up, and fewer policies will lapse when the capital market is down.

- In an up market, the value of the minimum guarantee diminishes as the account value is likely to exceed the minimum guarantee values. As such, surrendering the policy does not create much loss to the policyholder.
- On the other hand, a down market can result in the surrender value being less than the guarantee value causing the policy to be in-the-money. If the policyholder surrounds at this time then he or she can only get the reduced surrender value, forfeiting the added value from the guarantee rider. The result is that there is strong incentive for the policyholder to keep the in-the-money VA contract in force.

As the lapse assumption may impact significantly the profitability of GMAB riders, a rigorous modeling framework of the lapse rate is necessary for both pricing and hedging purpose. During the last decade, the literature on pricing and risk management of these guarantees has been evolving.

- Tradionnally the lapse behavior has been modeled by historical or backward looking statistical regressions which have empirically underestimated the risk due to the scarcity of extreme scenario samples for these new products and the inability to dynamically extrapolate the observed behavior to various market conditions.
- In contrast, a "rational" lapse strategy valuation is a prudent forward looking approach where policyholders lapse in a way that maximizes the net present value of the future cashflows, depending on key drivers. This reflects a potential extreme policyholder behavior, as experienced in the last market crash, with an initial immediate and sustained fall in lapses right after the crash, before an abrupt recovery consistent with the interest rates. In contrast, dynamic lapses modeling are usually unable to provide such empirical dynamics.

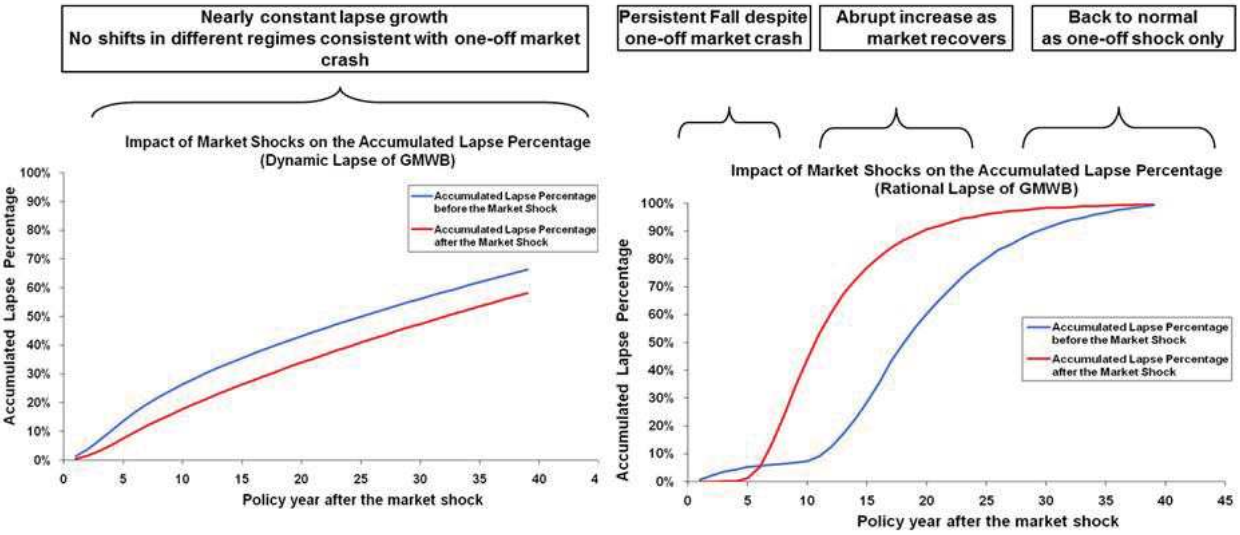

Fig. 1. Impact of market shocks on the accumulated lapse percentage in the case of dynamic and rational lapses for a GMWB

This approach is illustrated on a GMAB VA product using two alternatives numerical schemes (PDE and Monte Carlo), as the valuation of a Bermudan-style contingent claim for the insurer, where the contingency is closely related to equity market conditions and the interest rate level (see [15]).

The price evaluated by this approach can be interpreted as the fair value of the policy if all policyholders use the same rational lapse strategy, which is similar with the optimal early-exercise strategy of Bermudan options. However, as policyholders cannot be expected to lapse all at the same time, this rational lapse framework is slightly amended by introducing a proportion of lapses among the contract still active, which translates into the notion of "reasonable" lapse more consistent with empirics. Note that this is only an interpretation, and that the critical aim is to make sure the lapse risk can be hedged no matter which strategy the holders use.

The remainder of this paper is organized as follows. Firstly, in section 1 the GMAB policy is explained in full details. Section 2 introduces the modeling framework to evaluate the liability of GMAB policies in no-lapse assumption. The rational lapse strategy and critical lapse boundaries are studied in section 3. Section 4 addresses the pooling of lapse risks. In section 5 we implement two numerical methods, the PDE approach and the high-dimensional regression method (Monte Carlo, see [14]) to calculate the no-arbitrage price of GMABs in the Hull-White interest rate model. Numerical results of these two methods are shown in section 5 The final section concludes the paper.

1. Description of the contract

In practice, most GMAB policies are purchased in a lump sum. We assume that a single premium is paid at inception of the contract and denoted by $\breve{A}(0) =$ 100$ the initial account value at time zero after the upfront fees have been paid. The account value is invested in a portfolio consisting mainly of equities and bonds. At the end of each policy year t_i, the insurer deduct a charge fee $\overline{c}\,\breve{A}(t_i)$ on the account value, where $\overline{c} = 2\%$ is the annual charge rate. The life time of the policy is $T = 10$ years if there is neither early termination nor rollover.

For a contract that is held until the maturity, there is a guaranteed minimum return paid to the policyholder. We represent this guarantee to the policyholder as B, which is called the benefit base for insurers. In other words, at the maturity, the policyholder has the right to receive a cash payment equal to either $\breve{A}(T)$ or to the benefit base B. Consequently, at maturity, the value of the policy is $\max(\breve{A}(T), B)$.

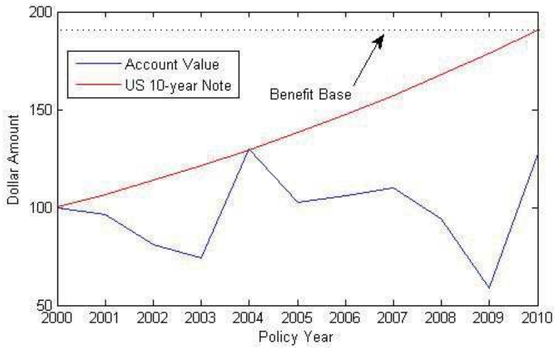

Fig. 2. The illustration of the account value over time compared with the US 10-year note

This payoff can be decomposed to the sum of the account value $\breve{A}(T)$ and a vanilla put $(B - \breve{A}(T))^+$ (the strike level is B). The benefit base B is fixed at inception, which is equal to $\breve{A}(0)(1+\overline{r})^T$, where $\overline{r}$ is the roll-up rate. In most cases, $\overline{r}$ is approximately equal to the yield of zero-coupon bonds maturing at T.

We assume that one GMAB policy is purchased in 2000 and hold until 2010, and the account value is invested in S&P 500 at inception. The roll-up rate is set at 6.67%, which is equal to the yield of US 10-year notes in January 2000. All other parameters are the same as those mentioned above. Figure 2 plots the account value $\breve{A}(t)$ over time. By comparing the net return of 10-year bonds with that of the roll-up GMAB rider, we can see that the roll-up benefit base can not only protect policyholders from catastrophes in stock market, but also from risks of the persistent decrease of the interest rate[1].

2. Valuation of a GMAB with zero lapse

Firstly we establish the general modeling framework to evaluate the liability of GMAB with zero lapse (European GMAB). From now on, we let $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t \leqslant T}, \mathbb{Q})$ denote a complete filtered probability space supporting two independent standard one dimensional Brownian motions W and $W^{\perp}$. Here $T > 0$ is a fixed time horizon. We assume that the filtration $\mathbb{F}$ is the completion of the rough filtration generated by $(W, W^{\perp})$, so that any martingale $(\mathbb{Q}, \mathbb{F})$-martingale can be represented as a stochastic integral with respect to $(W, W^{\perp})$.

During the last decade the literature on pricing Variable Annuities has evolved, but many evaluation approaches proposed (e.g. [5], [30]) are still based on the assumption of deterministic interest rates. Such an assumption is harmless in most situations since the interest-rates variability is usually negligible when compared to the variability observed in equity markets. While pricing a long-maturity securities such as VA guarantees, however, the volatile feature of interest rates can have stronger impacts on the liability of GMAB. In such case it is therefore advisable to use stochastic interest rate models.

In this paper, we assume that the short term interest rate $r = (r(t))_{t \geq 0}$ is driven by the one factor Hull and White model, and the underlying asset $S = (S(t))_{t \geq 0}$ in which the account value is invested follows a Black and Scholes type dynamics, namely:

$$\begin{cases} dS(t) = r(t)S(t)dt + \sigma S(t)dW(t) \\ dr(t) = a(\theta(t) - r(t))dt + \sigma_r dZ(t) \, . \\ Z := (1-\rho^2)^{\frac{1}{2}} W^{\perp} + \rho W \end{cases} \quad (1)$$

Here, a and σ_r are positive constants, θ is a deterministic Lebesgue-integrable function, σ is the instantaneous volatility of the asset return, and ρ is the correlation[2] between the account value and the interest rate. Note that the above financial market is complete whenever S and a zero-coupon bond with maturity T can be freely traded, and that $\mathbb{Q}$ is the only martingale (risk neutral) measure.

For the account value, a charge fee is deducted at a rate c continuously, where $c = -\log(1 - \overline{c})$. This means that $\breve{A}(t)$ evolves according to

$$d\breve{A}(t) = (r(t) - c))\breve{A}(t)dt + \sigma\breve{A}(t)dW(t). \quad (2)$$

Since $(r, \breve{A})$ is a Markov process, the European-style liability L^E of a single GMAB rider can be identified to a deterministic liability function ℓ^E by:

$$\begin{aligned} L^E(t) : \ell^E(t, r(t), \breve{A}(t)) &= \mathbf{E}^{\mathbb{Q}}[D_t^T \max(\breve{A}(T), B) \mid \mathcal{F}_t] = \\ &= \mathbf{E}^{\mathbb{Q}}[D_t^T(\breve{A}(T) + (B - \breve{A}(T))^+) \mid \mathcal{F}_t] \end{aligned} \quad (3)$$

where $D_{t_1}^{t_2}$ represents the stochastic discount factor between t_1 and t_2.

$$D_{t_1}^{t_2} := \exp\left(-\int_{t_1}^{t_2} r(s)ds\right).$$

[1] In fact, the US 10-year bond yield was 6.67% in January 2000, while it was 3.61% ten years later.

[2] For VAs, the correlation is often negative, as most portfolio contains fixed income assets, such as bonds.

Equation (3) shows that the European-style liability of GMAB riders can be considered as the sum of a forward contract of the account value ending at T and a vanilla put with the maturity T and the strike level B. In the Hull-White interest rate model, this liability value can be easily calculated analytically.

However, it does not always exist some closed formula of the liability value, especially when the early-lapse premiums are taken into account. Thus in practice, we need to use some numerical methods, such as PDE or Monte-Carlo based algorithms to evaluate the policies. For GMAB riders, it is sometimes more convenient to price the liability under the so-called forward measure $\mathbb{Q}^T$ (see [21]) rather than the risk-neutral measure $\mathbb{Q}$ (see [20]). Because in $\mathbb{Q}^T$, we can reduce the number of dimensions of the liability evaluation problem (3) from three to two and the pricing process can be significantly accelerated.

To facilitate the following study, we evaluate the GMAB riders in the forward measure $\mathbb{Q}^T$. Firstly, we introduce the forward value of $\breve{A}(t)$ at T observed at date t, denote by $F^T(t) = \breve{A}(t)/Z_t^T$ where Z_t^T is the price of a zero-coup bond maturing at T. Applying Ito's lemma to $F^T(t)$, we get the dynamics of the forward account value:

$$\frac{dF^T(t)}{F^T(t)} = (\nu^2 B_r^2(u) + \rho\nu\sigma B_r(u) - c)dt + \\ + \sigma dW(t) + \nu B_r(u) dZ(t),$$

where $u = T - t$ is the time to maturity and the function $B_r(u) = (1 - e^{-au})/a$. By doing the following transformations of the Brownian motions from $\mathbb{Q}$ to $\mathbb{Q}^T$:

$$\begin{aligned} dZ(t) &\to dZ^T(t) - \nu B_r(u)dt \\ dW(t) &\to dW^T(t) - \rho\nu B_r(u)dt \end{aligned} \tag{4}$$

we have that, under $\mathbb{Q}^T$, the dynamics of $F^T(t)$ can be written as:

$$\begin{aligned} dF^T(t) &= -cF^T(t)dt + \sigma F^T(t)dW^T(t) + \\ &+ \nu B_r(u)F^T(t)dZ^T(t) = \\ &= -cF^T(t)dt + \omega_u F^T(t)d\tilde{W}^T(t) \end{aligned} \tag{5}$$

where $\omega_u^2 = \sigma^2 + \nu^2 Br^2(u) + 2\rho\sigma\nu Br(u)$ and $\tilde{W}^T$ is a Brownian motion in $\mathbb{Q}^T$. The results above allow us to simplify the pricing problem of GMAB riders. Instead of computing the liability under risk-neutral measure $\mathbb{Q}$ (as in (3)), we evaluate the forward liability $\tilde{\ell}^E(t,f)$ in $\mathbb{Q}^T$:

$$\begin{aligned} \tilde{\ell}^E(t, F^T(t)) &:= \frac{\ell^E(t, r(t), \breve{A}(t))}{Z_t^T} = \\ &= \mathbf{E}^{\mathbb{Q}^T}[F^T(T) + (B - F^T(T))^+ \mid \mathcal{F}_t] \end{aligned} \tag{6}$$

Equation (5) and (6) show that the European-style forward liability $\tilde{\ell}^E$ can be evaluated by the following analytical formula:

$$\begin{aligned} \tilde{\ell}^E(t, F^T(t)) &= e^{-cu}F^T(t) + BN(-d_2) - e^{-cu}F^T(t)N(-d_1), \\ d_{1,2} &= \frac{\log(F^T(t)/B) - cu}{\Gamma} \pm \frac{\Gamma}{2} \end{aligned} \tag{7}$$

where $\Gamma = \sqrt{\int_0^{T-t} \omega_s^2 ds}$ and $N(\cdot)$ is the cumulative distribution function of the standard normal distribution.

3. Valuation a GMAB with rational lapse assumption

In the previous section, we have formulated the pricing issue of GMAB riders under the no-lapse assumption. If policyholders are not allowed to lapse contracts before maturity, the liability of GMAB riders can be calculated analytically by (7). However, in practice we can not assume the lapse rate to be zero or some other constant, as we observe that the lapse rate does change significantly in different market conditions (equity market and interest rate level) and this fluctuation of lapse rate has notable impacts on the liability value and insurer's hedging strategy.

As explained in the introduction, we consider the pricing problem of liabilities with lapse options as the valuation of a Bermudan-style contingent claim (see [12]). Because the rational lapse strategy discussed here are similar with the optimal early-exercise strategy of classic Bermudan options. We assume that the policyholder can lapse the contract at the end of each policy year, noted as t_i, $i = 1,2,...,T$. According to the definition, we have

$$\ell^B(T, r(T), \tilde{A}(T)) = \max(\tilde{A}(T), B)\ell^B(t_i -, r(t_i -), \tilde{A}(t_i -)) = ess\sup_{\tau_i \in \mathcal{T}_i} \mathbf{E}^{\mathbb{Q}}\left[D_i^{\tau_i}\breve{A}(\tau_i) + 1_{\{\tau_i = T\}} D_i^T \max(\breve{A}(T), B) \mid \mathcal{F}_{t_i}\right] \tag{8}$$

where $\mathcal{T}_i$ is the set of all stopping times taking values in $\{t_i, t_{i+1}, \dots T\}$ and τ_i denotes the stopping time of the rational lapse strategy since time t_i.

According to the assumption, the policyholder is not authorized to lapse the contract between two purchase anniversaries t_i and t_{i+1}, so the process

L^B of the liability should evolve in the same way as L^E for $t_i \le t < t_{i+1}$. Applying the fact that $r(t)$ and $\breve{A}(t)$ are all Markov processes, we have

$$\begin{aligned}&\forall i \leqslant n, \quad \ell^B(t_i, r(t_i), \breve{A}(t_i)) = \\ &= \mathbf{E}^{\mathbb{Q}}[D_i^{i+1} \ell^B(t_{i+1}-, r(t_{i+1}-), \breve{A}(t_{i+1}-)) \mid \mathcal{F}_{t_i}].\end{aligned} \tag{9}$$

Similarly with Bermudan-style options, at discrete time points t_i, the policyholder is supposed to compare the account value with the value of the liability to decide whether lapse or not. If the account value is bigger than the liability, policyholders surrender the contract and get back $\breve{A}(t_i)$. Otherwise, they continue to hold the policy. That is to say, at time t_i, the Bermudan-style liability should evolve as following

$$\begin{aligned}&\ell^B(t_i-, r(t_i-), \breve{A}(t_i-)) = \\ &= \max(\breve{A}(t_i), \ell^B(t_i, r(t_i), \breve{A}(t_i)))\end{aligned} \tag{10}$$

Equation (10) reflects the fact that the liability before the annuity payment is equal to the greater of the current account value $\breve{A}(t_i^-)$ and the value of continuation. To simplify the pricing process of the Bermudan-style liability, we can calculate the expectations under the forward measure $\mathbb{Q}^T$ instead of the risk neutral measure $\mathbb{Q}$. That is to say, we write (10) as

$$\begin{aligned}&\tilde{\ell}^B(t_i-, F^T(t_i-)) := \frac{\ell^B(t_i-, r(t_i-), \breve{A}(t_i-))}{Z_i^T} = \\ &= \max(F^T(t_i), \tilde{\ell}^B(t_i, F^T(t_i))) = \\ &= \max(F^T(t_i), \mathbf{E}^{\mathbb{Q}^T}\left[\tilde{\ell}^B(t_{i+1}-, F^T(t_{i+1}-)) F_{t_i}\right],\end{aligned} \tag{11}$$

where the boundary condition at maturity is

$$\tilde{\ell}^B(T, F^T(T)) = F^T(T) + (B - F^T(T))^+ \tag{12}$$

Another important issue related with the evaluation problem of the Bermudan-style liability is the determination of the rational lapse strategy to be followed. As the benefit base B is fixed atinception, according to (11) and (12), we have that $\tilde{\ell}^B$ (t, f) is a convex, nondecreasing function of f. In addition, it is also a positive function on $(t, f) \in [0, T) \times [0, \infty)$, for $\tilde{\ell}^B(t, f) > \tilde{\ell}^E(t, f) > 0.$ Finally, for the charge fee $c > 0$, (11) and (12) also imply that $\lim_{f \to \infty}(f - \tilde{\ell}^B(t_i, f)) > 0.$ It follows from the previous arguments that, for each $t \in \{t_0, ..., t_i, ..., t_{n+1}\}$, there exist a real number $f^*(t_i-)$,

$$\begin{aligned}&0 \le f \le f^*(t_i-) \Rightarrow \tilde{\ell}^B(t_i-, f) > f (\text{not lapse}), \\ &f \ge f^*(t_i-) \quad\;\; \Rightarrow \tilde{\ell}^B(t_i-, f) = f (\text{lapse}).\end{aligned} \tag{13}$$

In this paper, $f^*(t_i-)$ is referred to as the "critical forward account value" since the policy should be lapsed as soon as the forward account value increases to this level at time t_i. As it is shown by (13), thanks to the change of measure, the critical boundary here depends only on the forward account value, rather than on both the interest rate and the account value level. However, it is not always possible to do this kind of simplifications when we evaluate Bermudan-style options, because sometimes the intrinsic payoff (such as f for $\tilde{\ell}^B(t, f)$) is not a linear function of the underlying (e.g. American vanilla options).

The objective now is to evaluate the liability $\tilde{\ell}^B$ and the critical surface f^* of a single GMAB rider. Although many analytical approximations exist in academy literatures (see [30]), most of them are not sufficiently precise due to the long maturity property. In the present paper, we propose two numerical methods: PDE and Monte Carlo schemes (see Appendix C for the description of the numerical schemes and numerical tests for the results), to calculate both the Bermudan-style liability $\tilde{\ell}^B$ and the critical lapse surface. As we have mentioned at the beginning, the PDE method is precise for low-dimensional problems (< 3), while the Monte Carlo is more efficient when there are more than three dimensions in the pricing problem (e.g. multi-asset account value or stochastic volatility models).

4. Life insurance policy pool

The analysis above is focused on the rational lapse strategy and the no-arbitrage value of a single GMAB policy. The liability ℓ^B, which takes into account the lapse risks, allows the insurer to hedge the uncertain customer behavior no matter what the lapse strategy of the policyholder is. But in practice, the insurers often need to estimate the lapse risks of a pool of life insurance policies, and in this case, the lapse strategy can be represented by the frequency $p(t_i)$ of the policies that are early terminated at time ti (see [1]). Actually, the common sense and experience tells us that not all policyholders will lapse the contract at the same time, so we need to slightly change the function $p(t_i)$ to estimate the real lapse rate of a policy pool.

For a pool of GMAB policies, we denote by p (t_i) the proportion of lapses at date t_i among the contracts still active in the pool. According to the rational lapse strategy, we can express p (t_i) as a deterministic function f of the forward account value $F^T(t_i)$ that is $f(F^T(t_i)) = \mathbf{1}_{\{F^T(t_i) \geqslant f^*(t_i-)\}}$. This lapse function implies that once $F^T(t_i)$ touches $f^*(t_i-)$, all policyholders lapse the contract and otherwise everybody hold the policy.

Inspired by the mortgage prepayment models (see [2]) and evaluation approaches of surrender options for other life insurance products (see [1]), we assume that / is a nondecreasing piece-wise linear function of the variable $F^T(t_i)$. When $F^T(t_i) < f^*(t_i-)$, the lapse rate is not zero due to policyholders' personal circumstances (including liquidity and death), which is independent of financial considerations. These "irrational" lapses are analogues to noneconomic prepayment on low-rate mortgages. While when $F^T(t_i) \geq f^*(t_i-)$, some rational lapses never occur, and a reasonable specification of $p(t_i)$ may be illustrated as that in Figure 3.

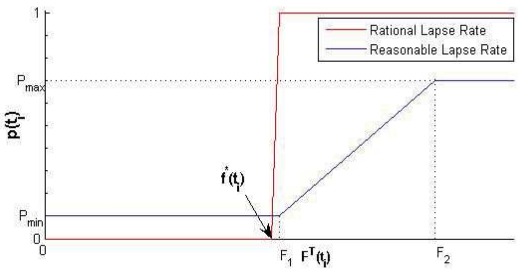

Fig. 3. Comparing the rational lapse function with the reasonable lapse function

The four parameters F_1, F_2, $P_{\min}$ and $P_{\max}$ are determined by insurers according to some empirical tests. To be consistent with the rational lapse assumption, F_1 should be very close to $f^*(t_i)$, and $P_{\max}$ should be set high enough (normally ≥ 50%). Under the reasonable lapse assumption, once the critical lapse level and the reasonable lapse function are determined, the GMAB liability can be simply evaluated as an European-style option (see [1]). However, it is worthy to mention that, unlike the rational lapse approach, the reasonable lapse assumption makes the insurer partially exposed to the risk of lapses in the future.

5. Numerical tests

In this section, we use two numerical methods (PDE and Monte Carlo) introduced above to evaluate the Bermudan-style liability of one standard GMAB policy. Our final results show not only the consistency between these two methods, but also the efficiency and precision of both methods. In addition, the option value of GMAB (defined later), the forward delta and the "critical boundary" found by these two methods are also compared.

For the tested policy, The account value is supposed to evolve according to Hull-White model, where the principle model inputs are listed in Table 1.

Table 1. Hull-White model inputs

σ	r_0	θ	a	$σ_r$	ρ
.2	0.02	0.02	0.03	0.01	0

The initial equilibrium short-rate curve $\theta(t)$ is supposed to be flat (θ constant) and the short rate at inception is denoted by r_0 All other parameters of product properties will be clarified later.

For simplicity, we also assume that the policyholder is alive at the maturity of GMAB policies. Although we are focused on the liability of a single policy in the following numerical tests, the methodology we propose here can be easily extended to evaluate a GMAB policy pool by adding up policies of different maturities with a proper weight indicated by mortality rate assumptions.

5.1. Bermudan-style GMAB liability. Firstly, we calculate the liability of a standard GMAB policy. The principle product parameters are listed in Table 2, where the charge fees rate is $c = 2\%$, the maturity is 10 years and the benefit base, fixed at inception, is 100$ for one policy. For simplicity, we assume that the policyholders are allowed to lapse the contract only at one specific date of each policy year.

Table 2. Product parameters of the GMAB policy

c	B	T	Lapse date frequency
%	100$	10 years	1/year

Figure 4 shows the forward Bermudan GMAB liability $\tilde{\ell}^B(t, f)$ computed by the PDE scheme for different forward account values through time. In addition, the intrinsic value of GMAB policies, which is equal to the instantaneous forward account value, is also recorded in Figure 4. It is worthy to mention that at the dates when lapses are allowed, once we have $\tilde{\ell}^B(t, f) = f$ for f big enough, policyholders should lapse the contract immediately. This phenomena is consistent with our intuitive, as the higher the account value is, the less the GMAB guarantees worth and the more likely that policyholders lapse the contract and get back the intrinsic value immediately.

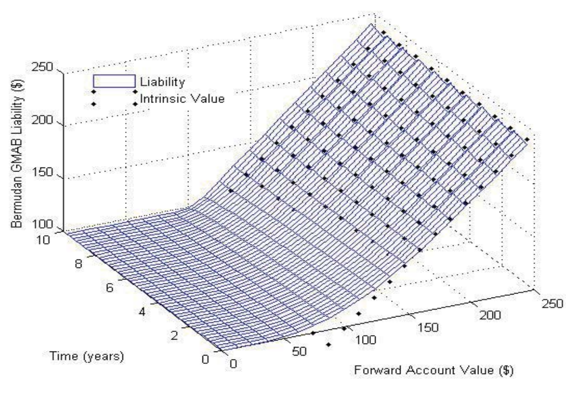

Fig. 4. The forward Bermudan liability ($\tilde{\ell}^B$) of a standard GMAB policy for different forward account values and time points, compared with the intrinsic value at all exercisable dates

To be further protected from potential lapse waves or other financial risks, the insurers can charge the policyholders an up-front fee, which is equal to $Z_0^T(\tilde{\ell}^B(0,f_0)-f_0)$ (the difference between the liability and the asset), to make sure the balance sheet is in equilibrium at inception (see [15]).

Figure 5 shows the early lapse premium (the difference between $\tilde{\ell}^B$ and $\tilde{\ell}^E$) calculated by the PDE Scheme. In this figure, we observe that the Bermudan liability is almost equal to the European-style liability when the account value falls to very low levels. Because in this case, the probability that policyholders lapse the contract before the maturity is extremely small. While when the forward account value increases to very high levels, the early lapse premium grows almost linearly with f. This is due to the fact that when $f \gg f^*(t)$, $\tilde{\ell}^B(t,f)=f$ and $\tilde{\ell}^E(t,f)\approx e^{-c(T-t)}f$. Finally we observe that, like other Bermudan contingent claims, the early lapse premium of GMAB policies reduces gradually to 0 at the expiration date.

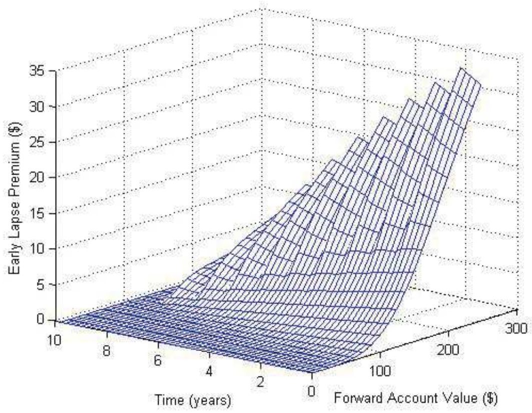

Fig. 5. The forward early lapse premium ($\tilde{\ell}^B - \tilde{\ell}^E$) of a standard GMAB policy for different forward account values and time points

5.2. Option value of the GMAB policy. Firstly, we define the forward option value, denoted as $\tilde{p}^B(t,f)$, of the Bermudan-style GMAB policy,

$$\tilde{p}^B(t,f) := \tilde{\ell}^B(t,f) - f. \tag{14}$$

The notation $\tilde{p}^B$ implies that the option value has many similar properties as a vanilla put[1] (see Appendix B). In fact, $\tilde{p}^B(t,f)$ can be simply interpreted as the difference between the liability $\tilde{\ell}^B(t,f)$ and the asset f of GMAB policy issuers. That is to say, $\tilde{p}^B(t,f)$ is the option that the insurers should replicate in practice.

[1] For European GMAB policies, we know that the option value is in fact a vanilla put (see (7)).

In Figure 6, we provide the numerical results obtained by the PDE scheme for different forward account values and different dates from the inception to the expiration of the policy. In this figure, we observe that the forward option value $\tilde{p}^B(t,f)$ evolves similarly as a vanilla put (see Appendix B). In addition, when the forward account value is significantly higher than the critical lapse boundary, the option value becomes negative between two discrete exercisable dates. This is due to the charge fees that policyholders are obliged to pay to insurers.

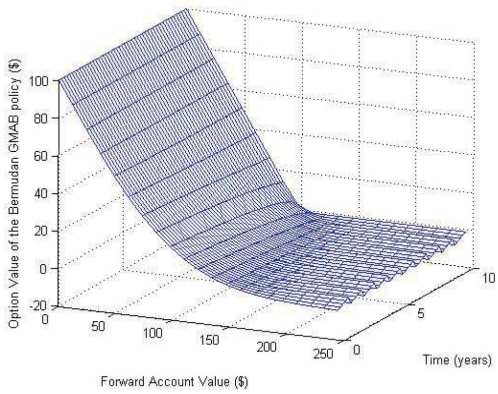

Fig. 6. The forward option value ($\tilde{p}$) of a standard GMAB policy for different forward account values and dates from the inception to the expiration of the policy

Figure 7 compares the numerical results of $\tilde{p}^B(0,f)$ computed by two methods: PDE and Monte Carlo. For the Monte Carlo method used here, we simulate 10,000 scenarios with the step length of 0.1 year. To evaluate $\tilde{p}^B(t,f)$ in this example, the average computing time of Monte Carlo method is about 1 to 2 seconds. Figure 6 shows that the prices calculated by Monte Carlo-$S1$ method is consistent with the results of PDE method.

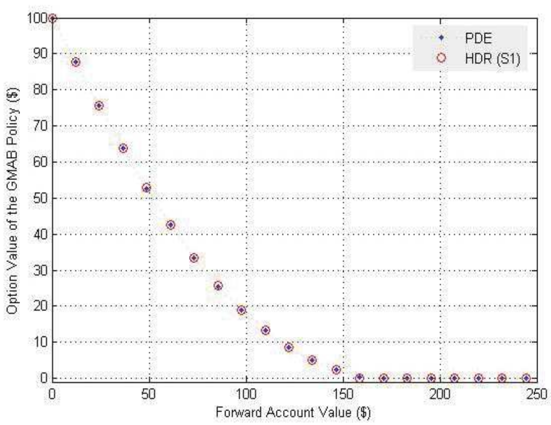

Fig. 7. Comparing the numerical results of the option value $\tilde{p}(0,f)$ computed by two different methods: PDE and Monte Carlo

We also compare the numerical results of the forward delta of the option value computed by PDE and Monte Carlo-S1 in Figure 8. We observe that the forward delta jumps up to 0 very quickly when f approaches to the critical boundary. This is easy to understand, as in this case all policyholders are supposed to lapse the contract and the insurers have no more need to hedge their liabilities. Fortunately, this difficulty of hedging lapse risks near the critical boundary can be partly overcome by diversifying the portfolio of GMAB policies (e.g. different maturities and benefit base levels).

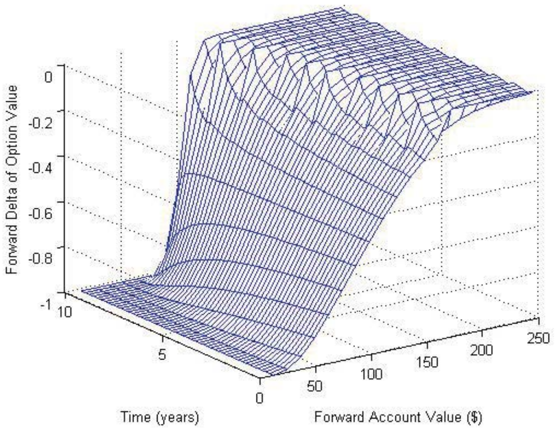

Fig. 8. The forward delta of $\tilde{p}(t, f)$ of a standard GMAB policy for different forward account values and dates from the inception to the expiration of the policy

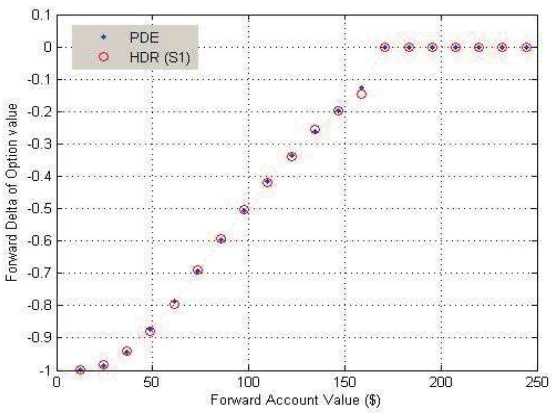

Fig. 9. Comparing the numerical results of the forward detla of the option value computed by two different methods: PDE and Monte Carlo-S1

Figure 10 compares the critical lapse boundary for different discrete dates computed by the two methods introduced above. We observe that the numerical results of Monte Carlo method becomes more and more instable from the expiration to the inception of the policy. This is because of the accumulation of pricing errors caused by linear regressions backward through time.

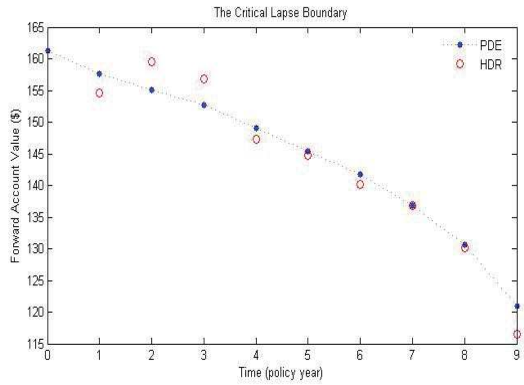

Fig. 10. Comparing the critical lapse boundary computed by two methods: PDE and Monte Carlo

In summary, we find that in our specific example here, the PDE method is faster and more precise, especially for f nearing the critical boundary, than Monte-Carlo based methods. In addition, this method can calculate the price and other important Greeks for different f and t at the same time. However, compared with the PDE method, the Monte Carlo method is much more flexible and easier to be implemented. Moreover, unlike PDE based methods, the Monte Carlo method can be extended to other high-dimensional problems, such as path-dependent payoffs, stochastic volatility models or basket account values (see [14], [15]).

Conclusion

In this paper, we introduce a framework to evaluate the liability of GMAB polices under rational lapse assumption. We study in full details not only the financial sensitivities, but also the rational lapse strategy of GMAB products in the stochastic interest rate model. Two numerical methods, the PDE and Monte Carlo, are implemented to price the policy and also to determine the critical lapse boundary. Moreover, we find a semi-analytical formula to approximate the lapse premium of the GMAB. Inspired by the rational lapse assumption, we finally introduce the reasonable lapse assumption to help insurers to measure the lapse risks of a policy pool.

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Appendix A. American-style GMAB

To the best of our knowledge, there is no closed formula to evaluate the Bermudan-style liability $\tilde{\ell}^B(t,f)$. However, we can use some semi-closed formulas to approximate the liability if we assume that the policyholders can lapse the contract at any time. In this case, we denote the GMAB liability as $\tilde{\ell}^B(t,f)$, which is in fact an American-style contingent claim.

In the past twenty years, many analytical approaches for evaluating American-style options in the Black-Scholes model are published, such as [8], [18], [10], [16], etc. However, most of them are not flexible for different payoff functions. In this paper, we find that the BAW and JZ approaches (see [8] and [18]) can be extended to estimate $\tilde{\ell}^B(t,f)$ in the one factor Hull-White model. Numerical tests show that these two approximations are efficient and precise for GMAB policies with 20 years maturity.

Firstly, we show how the BAW method can be applied directly to estimate the American-style liability $\tilde{\ell}^B(t,f)$. It is obvious that $\tilde{\ell}^B(t,f)$ is the solution of (29),

$$\frac{\partial \tilde{\ell}^A}{\partial t} - cf\frac{\partial \tilde{\ell}^A}{\partial f} + \frac{w_{T-t}^2 f^2}{2}\frac{\partial^2 \tilde{\ell}^A}{\partial f^2} = 0 \tag{15}$$

and is subject to the boundary conditions $\tilde{\ell}^A(t-,f) = \max(f, \tilde{\ell}^A(t,f))$ for $0 \le t \le T$. The key insight of BAW approximation (see [8]) is that if both American options and European options are solutions of (29), then the early exercise premium $\psi(t,f)$ of GMAB policies, which is equal to $\tilde{\ell}^A(t,f) - \tilde{\ell}^E(t,f)$, is also a solution of (29). Defining $\tau = T - t$ and changing the variable of ψ from t to τ, we have that $\psi(t,f)$ is the solution of the following equation,

$$-\frac{\partial \psi}{\partial \tau} - cf\frac{\partial \psi}{\partial f} + \frac{1}{2}\omega_\tau^2 f^2 \frac{\partial^2 \psi}{\partial f^2} = 0. \tag{16}$$

In practice, it is very difficult to find the solution of (16) analytically. So the authors of [8] developed an approximation method to estimate $\psi(t,f)$.

According to the BAW method, the early exercise premium can be approximated by the function $\overline{\psi}(\tau,f) = h(\tau)u(h,f)$, where $h(\tau) = 1 - e^{-g\tau}$ (numerical tests show that $g = -\log(Z_t^T)/\tau$ could be a good choice) and $u(h,f)$ is a function to determine. Replacing $\psi(t,f)$ by $\overline{\psi}(\tau,f)$ in (16) and neglecting the term $\partial u / \partial h$, we have:

$$-\frac{ge^{-g\tau}}{1-e^{-g\tau}}u - cf\frac{\partial u}{\partial f} + \frac{1}{2}\omega_\tau^2 f^2 \frac{\partial^2 u}{\partial f^2} = 0. \tag{17}$$

The general solution $u(h,f)$ of (17) is:

$$u(h,f) = A_1 f^{\lambda_1} + A_2 f^{\lambda_2},$$

where $\lambda_{1,2} = \dfrac{\omega_\tau^2 + 2c \pm \sqrt{(\omega_\tau^2 + 2c)^2 + 8ge^{-g\tau}\omega_\tau^2 / h(\tau)}}{2\omega_\tau^2}$.

As $\lambda_2 < 0$ while the early exercise premium is worthless when the asset price drops to zero, the coefficient A_2 must be zero. Thus when $f < f^*(t)$ at time t, the American-style liability can be approximated by:

$$\tilde{\ell}^A(t,f) \approx \tilde{\ell}^E(t,f) + h(\tau)A_1 f^{\lambda_1}. \tag{18}$$

It remains the coefficient A_1 and the critical forward account value $f^*(t)$ to find. In fact, (13) implies that at $f^*(t)$, $\tilde{\ell}^B(t,f)$ is equal to the forward account value, that is

$$f^*(t) = \tilde{\ell}^E(t, f^*(t)) + h(\tau)A_1 f^*(t)^{\lambda_1} \tag{19}$$

and the slope of the exercisable value, which is the forward account value, is set equal to the slope of $\tilde{\ell}^B(t,f)$ at $f^*(t)$, that is,

$$1 = \frac{\partial \tilde{\ell}^E(t,f)}{\partial f}\Big|_{f=f^*(t)} + h(\tau)\lambda_1 A_1 f^*(t)^{\lambda_1 - 1}. \tag{20}$$

Solving (19) and (20) by the algorithm of Newton-Raphson (see [8]), we can find both $f^*(t)$, and A_1 at time t.

The numerical tests show that the pricing error of BAW method is tiny if the forward account value f is not too small. However, when the GMAB policies are deep in the money, the BAW approximation becomes less precise. To improve the precision in this case, we extend the method developed by Ju and Zhong (JZ method, see [18]) to our specific evaluation problem here. In fact, the authors of [18] proposed to add a perturbation term to the function $\overline{\psi}(\tau,f)$ to improve the precision of the early exercise premium. This corrected function, denoted by $\overline{\psi_j}$, is defined as:

$$\overline{\psi_j}(\tau,f) := (1+\varepsilon(h,f))\overline{\psi}(\tau,f) = (1+\varepsilon(h,f))h(\tau)u(h,f), \tag{21}$$

where $\varepsilon(h,f)$ is a function to determine. Replacing ψ by $\overline{\psi_j}$ in (16) and applying (17), we obtain an equation for $\varepsilon(h,f)$ at time 0,

$$-\frac{\partial h}{\partial \tau}\frac{\partial u}{\partial h}(1+\varepsilon)-u\frac{\partial h}{\partial \tau}\frac{\partial \varepsilon}{\partial h}+(\omega_\tau^2 f^2\frac{\partial u}{\partial f}-cfu)\frac{\partial \varepsilon}{\partial f}+\frac{1}{2}\omega_\tau^2 f^2 u\frac{\partial^2 \varepsilon}{\partial f^2}=0. \tag{22}$$

After a series of approximations (see [18]), we get the corrected approximation to the American-style liability $\tilde{\ell}^B(t,f)$:

$$\tilde{\ell}^A(t,f)\approx\tilde{\ell}^E(t,f)+\frac{d}{1-bx^2-cx}\left(\frac{f}{f^*(t)}\right)^{\lambda_1}, \tag{23}$$

where $x=\log(f/f^*(t))$ and a, b, c and d are four parameters to be determined by (22). Figure 10 compares the American liability ℓ^A computed by (18) and (23) with the numerical results of PDE scheme, which is considered as the benchmarks here. We observe that the approximation methods are precise for a wide range of initial account values.

In addition, we find an empirical relationship between the Bermudan GMAB liabilities and the American ones, which can be simply written as

$$\ell^B(0,f)\approx\ell^A(0,f)-(\ell^A(0,f)-\ell^E(0,f))\frac{\Delta t}{T}, \tag{24}$$

where T is the maturity and Δt is the interval between two exercisable dates of Bermudan GMAB liabilities. Figure 11 verifies the empirical approximation (24) by the numerical results of PDE scheme. In fact, our numerical tests show that (24) is also applicable for other long-term Bermudan contingent claims (e.g. vanilla puts with maturities longer than 5 years).

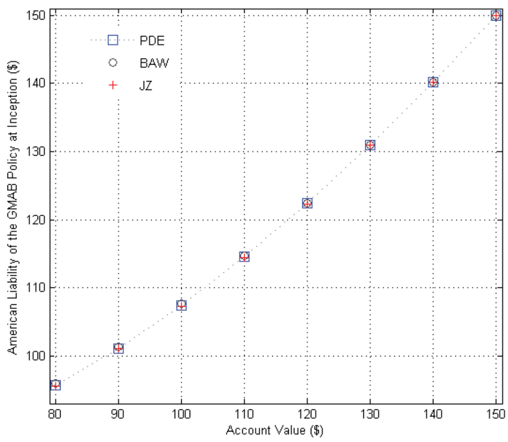

Fig. 11. Comparing the liability calculated by two approximation methods with the numerical results of PDE

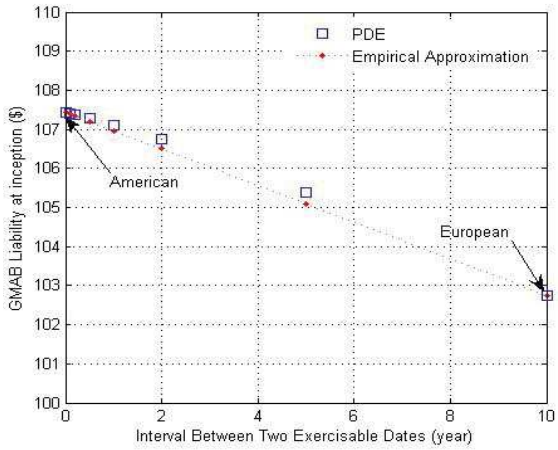

Fig. 12. Comparing the empirical approximation methods with the numerical results of PDE

Appendix B. Option value of GMAB policies

The forward option value $\tilde{p}^B(t,f)$ of GMAB policies, defined by (14), is what the insurers should replicate in practice once they write GMAB contracts. According to the definition, we can decompose $\tilde{p}^B(t,f)$ into two parts:

$$\tilde{p}^B(t,f)=\tilde{\ell}^B(t,f)-f=[\tilde{\ell}^B(t,f)-e^{-c\tau}f]-(1-e^{-c\tau})f \tag{25}$$

For simplicity, we define $\tilde{q}^B(t,f)=\tilde{\ell}^B(t,f)-e^{-c\tau}f$. Applying (11), it is easy to verify that $\tilde{q}^B(t_i-,F^T(t_i-))$ evolves as

$$\begin{aligned}\tilde{q}^B(t_i-,F^T(t_i-))&=\max((1-e^{-c(T-t_i)})F^T(t_i),\ \tilde{q}^B(t_i,F^T(t_i)))=\\&=\max((1-e^{-c(T-t_i)})F^T(t_i),\ \mathbf{E}^{\mathbb{Q}^T}[\tilde{q}^B(t_{i+1}-,F^T(t_{i+1}-))\,|\,\mathcal{F}_{t_i}])\end{aligned} \tag{26}$$

and at the maturity, we have $\tilde{q}^B(T,F^T(T))=(B-F^T(T))^+$. Therefore, we can interpret $\tilde{q}^B(t,f)$ as a Bermudan put option with the exercisable value $(1-e^{-c\tau})f$ at dates t_i. Some insurers call $\tilde{q}^B(t,f)$ as the forward value of claims, and $(1-e^{-c\tau})f$ as the forward value of charges, for this term is in fact the expectation of forward charge fees insurers can receive if the policyholder holds the contract to the maturity. According to (25), the forward option value $\tilde{p}^B(t,f)$ is equal to the difference between the forward value of claims and the forward value of charges.

Appendix C. Numerical schemes

It follows from the definition of the forward Bermudan-style liability $\tilde{L}^B(t)$ with an optimal stopping time $\tau \in \{t_1, t_2, \cdots, T\}$, that the process of the forward liability $\tilde{L}^B(t)$ satisfies the backward programming equation, for $0 \leq t_i \leq T$

$$\tilde{L}^B(t_i -) = \max\{F^T(t_i), \mathbf{E}^{\mathbb{Q}^T}[\tilde{L}^B(t_{i+1}-) \mid \mathcal{F}_{t_i -}]\} \tag{27}$$

and at the maturity, we have $\tilde{L}^B(T) = \max(B, F^T(T))$.

Thanks to the martingale property of $\tilde{L}^B$ on the interval $[t, \hat{\tau}_i)$, we have for $0 \leq t_i \leq T$,

$$\tilde{L}^B(t_i -) = \mathbf{E}^{\mathbb{Q}^T}[F^T(\hat{\tau}_i) + \mathbf{1}_{\{\hat{\tau}_i = T\}} \max(B, F^T(T)) \mid \mathcal{F}_{t_i}] \tag{28}$$

where the optimal stopping time $\hat{\tau}_i$ is defined as: $\hat{\tau}_i := \inf\{t_j \geqslant t_i : \tilde{L}^B(t_j -) = F^T(t_j -)\}$.

To the best of our knowledge, it is difficult to find precise analytical formulas to evaluate the Bermudan-style contingent claims in practice. In this paper, we extend the traditional semi-analytical methods (see [8] [18]) to estimate the spot Bermudian-style liability $\tilde{\ell}^B(t, f)$ of GMAB polices (see Appendix A). However, the approximation method introduced here is not as flexible as numerical approaches, especially for high dimensional problems. Thus in most cases, we need to use numerical methods, such as PDE and Monte Carlo, to calculate the Bermudan liability $\tilde{\ell}^B(t, f)$.

1. PDE scheme[1]**.** In this paper, we transform the evaluation problem (11) of Bermudan-style liability $\tilde{\ell}^B(t, f)$ into a free-boundary partial differential equation, for which $\tilde{\ell}^B(t, f)$ is the solution. For GMAB policies, the Bermudan-style liability $\tilde{\ell}^B(t, f)$ is represented as a function of two variables: the time t and the forward account value f. Applying Itô's lemma and the martingale representation theorem together, we know that the liability $\tilde{\ell}^B(t, f)$ is the solution of a one dimensional PDE. By adding the free-boundary constraint implied by equation (11) to this PDE, we have

$$\frac{\partial \tilde{\ell}^B}{\partial t} - cf \frac{\partial \tilde{\ell}^B}{\partial f} + \frac{w_{T-t}^2 f^2}{2} \frac{\partial^2 \tilde{\ell}^B}{\partial f^2} = 0 \tag{29}$$

on $\{(t, f) : t_{i-1} \leqslant t < t_i, f > 0\}$, subject to the boundary conditions at time points $0 < t_i < t_{n+1}$

$$\tilde{\ell}^B(t_i -, f) = \max(f, \tilde{\ell}^B(t_i, f)) \tag{30}$$

and at the maturity T, we have

$$\tilde{\ell}^B(T, f) = \max(f, B). \tag{31}$$

On each of the intervals $[t_{i-1}, t_i)$, the PDE (29) can be calculated numerically by using the Crank-Nicolson method (see [19]) for $f \in [0, F)$, where F is the upside boundary of the numerical solution. While at discrete time points t_i, the critical lapse surface $f^*(t_i)$ can be easily found by the free-boundary constraint indicated in (30). On the boundary, we impose the zero-convexity conditions[2]:

$$\tilde{\ell}^B(t, f)|_{f=0} = B; \qquad \frac{\partial^2 \tilde{\ell}^B}{\partial f^2}|_{f=F} = 0.$$

In fact, according to [4], the precision of the final solution is not very sensible to the error on boundaries if the solution domain of parabolic equation is large enough. So in most cases, the practitioner can choose other boundary conditions instead of those we propose here[3].

2. Monte-Carlo Scheme. The liability $\tilde{L}^B(0)$ is estimated as the conditional expected value of the forward liability based on Monte-Carlo simulation.

The forward liability satisfies two conditions:

[1] In this numerical test, the discrete time step of the PDE scheme is 0.01 year.
[2] This assumption is based on the fact that the gamma of the liability is small on the boundary.
[3] In the specific case here, the first or second order derivative boundary condition is preferred to the Dirichlet condition. As the latter could lead to significant errors on the boundary.

- the backward programming equation:

$$\begin{cases} \tilde{L}^B(t_i-) = \max\{F^T(t_i), \mathbf{E}^{\mathbb{Q}^T}[\tilde{L}^B(t_{i+1}-) \mid \mathcal{F}_{t_i-}]\} \\ \tilde{L}^B(T) = \max(B, F^T(T)) \end{cases} \tag{32}$$

- the martingale property of $\tilde{L}^B$ on each $[t, \tilde{\tau}_i)$ traduced by:

$$\begin{cases} \tilde{L}^B(t_i-) = \mathbf{E}^{\mathbb{Q}^T}[F^T(\hat{\tau}_i) + \mathbf{1}_{\{\hat{\tau}_i = T\}} \max(B, F^T(T)) \mid \mathcal{F}_{t_i}] \\ \hat{\tau}_i := \inf\{t_j \geqslant t_i : \tilde{L}^B(t_j-) = F^T(t_j-)\} \end{cases} \tag{33}$$

As pointed out in [14], these equations ((27) and (33)) lead to two algorithms, referred to as S1 and S2 hereafter.

The first algorithm S1 computes the optimal stopping time to lapse in three steps:

1. Simulate N discrete scenarios of the forward account value, denoted as $F^{T(k)}$ ($0 \le i \le n+1$ and $0 < k \le N$), according to (5).
2. Set the forward Bermudan-style liability at maturity for each scenario: $\tilde{L}_{[1]}^{B(k)}(T) = F^T(T)$.
3. Apply (27) from t_n to t_0. For $i = n$ to 0:

$$\text{if } F^{T(k)}(t_i-) < B : \tilde{L}_{[1]}^{B(k)}(t_i-) = \tilde{L}_{[1]}^{B(k)}(t_{i+1}-),$$

$$\text{if } F^{T(k)}(t_i-) \geqslant B : \tilde{L}_{[1]}^{B(k)}(t_i-) = \max\{F^{T(k)}(t_i-), \tilde{\mathbf{E}}^{\mathbb{Q}^T}[\tilde{L}_{[1]}^{B}(t_{i+1}-) \mid F^{T(k)}(t_i-)]\}.$$

From step 3 of scheme S1, we can identify the estimated rational lapse time $\tilde{\tau}_0^{(k)}$ as the first time for the k-th scenario where the liability equals the account value. Once $\tilde{\tau}_0^{(k)}$ is recorded for each path, we can estimate the Bermudan-style liability by scheme S2 where we regress the cash flows on a set of basis functions.

This latest computes the corresponding liability following four steps:

1. Simulation: Use the same N simulated scenarios as in S1.
2. Initialization: Set the rational lapse time $\tilde{\tau}_0^{(k)} = t_{n+1}$, for $0 < k \le N$.
3. Backward induction: For $i = n$ to 0, $\tilde{\tau}_i^{(k)} = i\mathbf{1}_{\{(k)\in\mathfrak{L}_i\}} + \tilde{\tau}_{i+1}^{(k)}\mathbf{1}_{\{(k)\in\mathfrak{L}_i^c\}}$. (where $\mathfrak{L}_i := \{(k) : L_{[1]}^{B(k)}(t_i-) = F^{T(k)}(t_i-)\}$ and $\mathfrak{L}_i^c\{(k) : L_{[1]}^{B(k)}(t_i-) > F^{T(k)}(t_i-)\}$ its complement).
4. Price estimator at 0: $\tilde{L}_{[2]}^B(0) := \frac{1}{N}\sum_{k=1}^{N}[F^{T(k)}(\tilde{\tau}_0^{(k)}) + \mathbf{1}_{\{\tilde{\tau}_0^{(k)}=T\}} \max(B, F^{T(k)}(T))]$.

In [14], the authors find the following relation with the two estimators $\tilde{L}_{[2]}^B$ and $\tilde{L}_{[1]}^B$ computed above:

$$\mathbf{E}[\tilde{L}_{[2]}^B(0)] \leqslant \tilde{L}^B(0) \leqslant \mathbf{E}[\tilde{L}_{[1]}^B(0)]. \tag{34}$$

In the numerical tests in the section below, we calculate both $\tilde{L}_{[1]}^B(0)$ and $\tilde{L}_{[2]}^B(0)$ to construct confidence intervals of the $[\tilde{L}_{[2]}^B(0), \tilde{L}_{[1]}^B(0)]$ for the true value $\tilde{L}^B(0)$.

Appendix D. Linear regression vs. global polynomial regression

We now introduce the scheme used to calculate the conditional expected value of continuation for scenarios such that $F^{T(k)}(t_i-) > B$. Here we use the local linear regression approach proposed in [14] to calculate this value, as opposed to the global polynomial regression method developed in [27]. The reason for this is that the latter can lead to some instability in the regression process for high dimensional and long maturity problems (see [14]).

For our specific problem, we have only one dimension: the forward account value F^T. The idea is to use, at each time step t_i, a set of functions ψ_d having local hypercube supports D_l, where the space is cut into I regions, l=1 to I and $\{D_d\}$ is a partition of $[\min_{\{k=1,N\}} F^{T(k)}(t_i), \max_{\{k=1,N\}} F^{T(k)}(t_i)]$. The index $(\cdot)^{(k)}$ denotes the k-th simulated scenario. On each support D_l, we define a linear function Ψ_l with 2 degrees of freedom, which are represented by a constant and F^T.

Our goal now is to regress the future cash flow of liability on the function Ψ_l to estimate the relevant conditional expectation. The two regression basis of Ψ_l, noted as (ψ_l^0, ψ_l^1), will be clarified later[1].

For simplicity, we define the function $G^N(t_i, F^T(t_i))$ as the conditional expectation at time t_i, we have:

$$G_i^{N(k)} : G^N(t_i-, F^{T(k)}(t_i-)) = \tilde{\mathbf{E}}^{\mathbb{Q}^T}[L_{[1]}^B(t_{i+1}-, F^T(t_{i+1}-)) \mid F^{T(k)}(t_i-)],$$

where $F_i^{N(k)}$ is the conditional expectation associated with the k-th path at time t_i. In the context of S1, the numerical procedure to calculate $G_i^{N(k)}$ reads as follows.

Scheme S_C: estimator of $G_i^{N(k)} (0 \leqslant i \leqslant n)$ with regression:

1. At time t_i, realize a quick-sort of $F^{T(k)}(t_i)$ for N scenarios and identify the support D_l of the functions Ψ_l so that each support contains approximately the same number of scenarios.
2. For each scenario $0 < k \leq N$, set the three regression basis of $\Psi_l : (\psi_l^0, \psi_l^1)$, where $\psi_l^0(\cdot) = 1$, $\psi_l^1(F^{T(k)}(t_i-)) = F^{T(k)}(t_i-)$.
3. On each support D_l, regress $\{L_{[1]}^{B(k)}(t_{i+1}-)\}_{k \leqslant N}$ on Ψ_l. In other words, for $\forall l$, we calculate the coefficients (α_l^0, α_l^1) that minimize $\sum_{k=1}^{N} | L_{[1]}^{B(k)}(t_{i+1}-) - \sum_{m=0}^{1} \alpha_l^m \psi_l^m(\cdot^{(k)}) |^2$, and set $G_i^{N(k)} = \sum_{m=0}^{1} \alpha_l^m \psi_l^m(F^{T(k)}(t_i^-))$.

[1] The two regression basis correspond to the constant and the forward account value.