

# “The uncertainty risk driver within a life annuity context: an overview”

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## The uncertainty risk driver within a life annuity context: an overview

### Abstract

The paper analyzes the longevity effects on the portfolio valuations. This is a relevant topic, in particular from the perspective of insurers/sponsors of pension funds. The models chosen for actuarial calculations have to capture the survival trend and to project its forecasted future improvements. The uncertainty in the choice is a huge concern and constitutes a relevant systematic risk driver itself, called uncertainty risk therein. Aim of the paper is to measure the uncertainty risk and to show its trend in several contexts, meaningful in portfolio valuations.

To this purpose the authors provide a suitable risk index and apply it in three different valuations: the initial value of an immediate life annuity portfolio; the fund of a pension annuity portfolio; the surplus of a portfolio consisting of deferred life annuities. Some graphs illustrate the results.

**Keywords:** conditional expectation, pension annuities, stochastic survival functions.

**JEL Classification:** C53, G17, G22, G32.

### Introduction

The increasing expansion of life expectancy in developed countries is one of the most insidious risk drivers in various contexts of the contemporary economic system, especially with regard to retirement income security and intergenerational issues relating to the running of pension funds (people in retirement vs. people in the workforce).

From the perspective of insurers/sponsors of pension funds, *longevity risk* is the systematic demographic risk linked with the prospect of facing future payments for longer durations than the expected ones (as the life expectancy of policy holders systematically increases). It is the downside of the demographic risk, opposing the *insurance risk*, which arises from accidental deviations of the mortality phenomenon itself.

In particular, managing pension funds must face the devastating combined effects between longevity risk and financial risk drivers; the last ones are due to both the volatility of return on investments and interest rate risk connected to lowering interest rates. This is more pronounced in the current context of global economic crisis, with regard to both social security and individual pension plans.

The choice of an appropriate management tool of the longevity risk is quite not simple and all the various ongoing solutions display strengths and weaknesses. Among the various solutions we recall traditional methods consisting of balancing hedging techniques, where compensating effects between opposite demographic risks are reached; reinsurance is another answer to the problem, even if it involves

cost-related and high interest risk correlation. Moreover transferring demographic risk to capital markets (LTR, i.e. Longevity Risk Transfer) is another possible key, even if a complete regulatory and structural arrangement of markets arising therefrom is still a remote goal.

Further alternatives lie in constructing survival-indexed contracts and the recent actuarial literature proposes several suggestions in those terms (cf. Richter et al., 2011; Piggot et al., 2005; Denuit et al., 2011; Maurer et al., 2013; D'Amato et al., 2013).

Longevity risk essentially takes shapes by two aspects, called, respectively, *rectangularization* and *expansion*.

The first one consists of an increasing concentration of deaths around the mode of the death curve, at adult ages; as consequence, the shape of the survival function tends to assume a rectangular configuration, so the term rectangularization is used to explain such phenomenon.

The second aspect, called expansion, by virtue of the previous one, concerns the mode of the curve of deaths, which tends to the ultimate age, itself moving towards very old ages.

It is evident that the reliability of mortality tables, which capture the survival trend and project its forecasted future improvements, is a huge concern. Within portfolios of products with longevity risk exposure, the choice of adequate *projected* mortality tables constitutes a relevant systematic risk driver, i.e. a model risk due to the uncertainty of the proper choice.

In Pitacco (2007) the risk of systematic deviations due to such choice is set in the context of parameter risk/table risk, linked up to the models for projecting mortality and their parameters. So the expression *uncertainty risk* can be properly

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used for defining the risk connected to all what concerns the systematic deviations of the demographic technical bases.

In this paper we will deep the uncertainty risk, analyzing it by means of a suitable risk measure. We will give particular attention to the uncertainty arising from the choice of incorrect demographic tables, so the conditional variance is the proper tool for measuring this specific risk. For this reason we are not interested in other forms of risk quantification (quantiles, Tail VaR and so on) (cf. also Coppola et al., 2011).

We will quantify the uncertainty risk in several contexts, meaningful in portfolio valuations. Under specified hypotheses for demographic and financial assessments, we will show the behavior of the uncertainty risk index. Within this framework, we will provide a feasible way to compare the risk trends with the specific aim of building hedging strategies.

The layout of the paper is the following: Section 1 we provide a measure of the uncertainty risk and extend results we already obtained (cf. Coppola et al., 2002; 2008 and 2011; Di Lorenzo et al., 2002). In Section 3 we apply the demographic risk measure proposed in Section 2 to (a) the initial value of an immediate life annuity portfolio; (b) a pension annuity portfolio fund; (c) the surplus of a portfolio consisting of deferred life annuities. In all the three cases the impact of the uncertainty risk is quantified and discussed by means of suitable numerical examples. Some concluding remarks close the paper.

In Appendix a brief outline of the survival functions used in the applications is presented.

### 1. Systematic risk indexes

Let us consider a portfolio of  $c$  homogeneous variable annuities, where each policy is issued to each of  $c$  lives aged  $x$  and  $a_h$  ( $h \geq 1$ ) is the variable installment payable, for instance, at the end of each year  $h$ , while  $(x)$  survives. Let indicate by  $K$  the random variable representing the curtate-future-lifetime of the  $i$ -th life insured and by  $Z_i$  the random present value of the annuity for the  $i$ -th life insured:

$$Z_i = \sum_{h=1, \dots, K_i^{(x)}} a_h v(0, h)$$

$v(0, h)$  being the stochastic present value at time  $t = 0$  of one monetary unit at time  $t = h$ .  $Z_i$  is defined to be 0 if  $K_i^{(x)}$  is 0.

Let us assume the following set of hypotheses (cf. Coppola et al., 2000 and 2002):

1. the random variables  $K_i^{(x)}$  are independent and identically distributed;
2. the random variables  $Z_i$  are independent and identically distributed, conditioning on the knowledge of  $(v(0, s))_{s=1, 2, \dots}$
3. the random variables  $K_i^{(x)}$  and  $v(0, s)$  are mutually independent.

Moreover let  $T$  be the random survival function from which the projected life tables are inferred,

$Z(c) = \sum_{i=1}^c Z_i$  the portfolio value and  $\frac{Z(c)}{c}$  the average cost per policy.

In the following we synthetically denote the demographic uncertainty risk by uncertainty risk.

We recall the risk filters provided by Coppola et al. in (2002):

**Definition 1A.**  $UR = V \left[ E \left[ \frac{Z(c)}{c} \mid T \right] \right]$  is a measure of the uncertainty risk.

**Definition 2.**  $IR = E \left[ V \left[ \frac{Z(c)}{c} \mid T \right] \right]$  is a measure of the investment risk.

Now we give another formulation of the overall systematic risk filter. Differently from Coppola et al. (2002), here we study the variance when the number of policies tend to infinity.

The following result holds:

**Proposition 1.** *Under the hypotheses (1), (2), (3), the limiting value of the variance of the average cost per policy, when the number of contracts tend to infinity, is a measure of the systematic risks.*

**Proof:**

Since

$$E[Z_i] = E \left[ E[Z_i \mid T] \right] = \sum_{h=1}^{\infty} a_h E[{}_h p_x] E[v(0, h)],$$

we obtain

$$E[Z(c)] = c \sum_{h=1}^{\infty} a_h E[{}_h p_x] E[v(0, h)]. \tag{1}$$

Moreover

$$E[Z_i^2] = E[E[Z_i^2|T]] = \sum_{h=1}^{\infty} a_h^2 E[_h p_x] E[v^2(0, h)] + \sum_{h=2}^{\infty} E[_h p_x] \left( \sum_{r=1}^{h-1} 2a_h a_r E[v(0, h)v(0, r)] E[_r p_x] \right),$$

hence:

$$E[Z^2(c)] = E\left[\sum_{i=1}^c Z_i^2 + \sum_{\substack{i,j=1 \\ i \neq j}}^c Z_i Z_j\right] = cE[Z_i^2] + c(c-1) \sum_{h=1}^{\infty} \sum_{k=1}^{\infty} a_h a_k E[_h p_x k p_x] E[v(0, h)v(0, k)]. \tag{2}$$

Let us consider the average cost per policy  $\frac{Z(c)}{c}$ ; by virtue of the pooling nature of the insurance risk, its impact can be controlled by increasing the number of policies.

We observe that:

$$V\left[\frac{Z(c)}{c}\right] = \frac{1}{c} E[Z_i^2] + \left(1 - \frac{1}{c}\right) E[Z_i Z_j] - (E[Z_i])^2 \tag{3}$$

and

$$\lim_{c \rightarrow \infty} V\left[\frac{Z(c)}{c}\right] = E[Z_i^2] - (E[Z_i])^2 = \sum_{i,j=1}^{\infty} E[_i p_x j p_x] a_i a_j E[v(0, i)v(0, j)] - \sum_{i,j=1}^{\infty} a_i a_j E[_i p_x] E[_j p_x] E[v(0, i)] E[v(0, j)]. \tag{4}$$

Now we observe that the sum of the two indexes introduced in Definitions 1A e 2A coincides with the result in equation (4). In fact:

$$\begin{aligned} UR + IR &= \sum_{i,j=1}^{\infty} E[_i p_x j p_x] a_i a_j E[v(0, i)v(0, j)] - \sum_{i,j=1}^{\infty} a_i a_j E[_i p_x] E[_j p_x] E[v(0, i)] E[v(0, j)] + \\ &+ \sum_{i,j=1}^{\infty} a_i a_j E[_i p_x j p_x] \text{cov}(v(0, i)v(0, j)) = \sum_{i,j=1}^{\infty} a_i a_j E[_i p_x j p_x] E[v(0, i)v(0, j)] - \\ &- \sum_{i,j=1}^{\infty} \sum_{i,j=1}^{\infty} a_i a_j E[_i p_x] E[_j p_x] E[v(0, i)] E[v(0, j)]. \end{aligned}$$

Finally, by comparing to the last side of formula (4):

$$UR + IR = \lim_{c \rightarrow \infty} V\left[\frac{Z(c)}{c}\right]$$

*QED*

The previous result means that the risk of very large portfolios is the systematic risk in the two forms of uncertainty and investment risks. This is consistent with what is stated in Coppola et al. (2000); in that paper the dependence on the number of policies came to light from the formulas of each risk measure and determined its behavior.

The preceding result can be easily extended to a portfolio of deferred life annuities.

## 2. The uncertainty risk: applications

The annuity contractual structure, both in immediate and deferred cases, suffers a complex sensibility to the uncertainty of the main variables involved in all the financial evaluations. In particular, focusing on the mortality/longevity risk, a wide literature and several evidences show how its impact varies accordingly to the ages and to the specific

contractual period. For instance, in the case of pension annuities, during the working period, when the annuitants pay the premiums, the insurer suffers the mortality risk (higher mortality means less premium payments) but probably this risk will not be so relevant in light of the young ages of the payers. On the contrary the longevity risk, particularly in its systematic and unremovable risk component (sometimes called *aggregated longevity risk*, cf. Olivieri et al., 2011), becomes important during the retirement period. This importance is not homogeneously spread during the retirement period, revealing age intervals in which it is particularly strong. In the preceding section we measured the importance of the impact of the aggregated longevity risk by means of the specific risk measure we indicated as *Uncertainty Risk*, and we think that the information provided by the description of this risk filter referred to a portfolio of policies and

studied as function of the time and of the age at issue of the pensioners/annuitants, can provide very useful indication to the insurer for building correct hedging strategies. These activities are realized in different forms: one of the most common is the buy-out plan, with which the pension plan transfers all the risks, including the longevity risk, to the insurer/reinsurer (cf. Blake et al., 2013). An alternative is a buy-in strategy: for example it can be specifically set for hedging the systematic longevity risk, aiming to a de-risking activity. Besides, other forms of hedging were explored. Initially (about ten years ago) a variety of bonds issued to the longevity hedging aim diffused. Often they suffered an insufficient demand, were characterized by a complex structure not easily correctly designable (think for example to the basis risk impact) and gave rise to a rigid and not liquid market. Successively new kinds of derivatives proliferated (i.e. q forward, longevity swaps). An interesting and wide overview of the different hedging strategies developed in the market are reported in Blake et al. (2013), as a proof of the great interest in this topic.

In the following subsections we apply the risk measuring approach proposed in the paper to a life annuity portfolio, specifically to its initial consistency, the portfolio fund and the portfolio surplus. In all the applications, aiming to purely illustrative purposes, we assume that the instantaneous rate of return on investment is described by a Vasicek process:

$$dr(t) = \alpha[\gamma - r(t)]dt + \sigma dW(t)$$

with  $\alpha = 0.07, \gamma = 0.02, \sigma = 0.04$  and initial position  $r(0)=0.025$ .

Under this stochastic hypothesis,  $v(t, j) = e^{-\int_t^j r(s)ds}$  is lognormally distributed.

Moreover the models we consider for depicting the future survival are the standard Lee Carter model (LC) (Lee and Carter, 1992), the Booth, Maindonald and Smith model (BMS) (Booth et al., 2002) and the Cairns, Blake and Dowd model (CBD) (Cairns et al., 2006). Measuring the impact of the uncertainty risk, as previously defined, is strongly affected by the reliability degree the insurer assigned to the demographic modes. In this example we assume to assign to LC and to BMS the reliability degree 0.4, to CBD 0.2. The study is referred to Italian male population. The most projected model is the CBD, followed by BMS and then LC (cf. D'Amato et al., 2015).

In Appendix we briefly recall the three survival models.

We emphasize that the choice of all the aforementioned financial and demographic processes is intended to provide an example; such choice can be modified depending on the application requirements.

**2.1. Application to the portfolio initial value.**

In this subsection we will show some graphical evidences of the Uncertainty Risk Measure. Our first application concerns the initial value of an immediate life annuity portfolio. In Figure 1 we consider the UR proposed by the Definition 1A

referred to the average cost per policy  $\frac{Z(c)}{c}$  and

show its trend when the age at issue varies from 20 to 80 and the time from 0 to 80. The figure shows that the Uncertainty Risk is characterized by a high concentration around the age 70, pointing out the existence of one risky age interval. This circumstance highlights that the older the insured is, the higher the initial UR is. High concentration is generally perceived as a good new from the hedging point of view, allowing for targeted and localized strategies. Moreover the younger the annuitant is at issue, the higher the UR in the risky age interval is.

The same trend is clearly shown in Figures 2 and 3 referred to the portfolio value  $Z(c)$ , respectively with the initial consistency of 500 and 10000 homogeneous policies, under the same hypothesis.

**2.2. Application to the portfolio fund.**

This second application regards a pension annuity portfolio fund of  $c$  policies, to which we adapt the risk measure in Definition 1A. The policies are issued on lives aged  $x$  and consist of a sequence of premiums  $P$  paid at the beginning of each contract term in case of life, till the retirement age occurring after  $n$  years. Then the insurer will pay a sequence of constant benefits  $R$  at the beginning of each contract term in case of life. The premiums are deposited in a fund earning interest term by term (cf. Olivieri et al., 2003).

If  $F(k)$  is the portfolio fund at time  $k$ ,  $N_k$  the number of survivors at time  $k$  belonging to the initial cohort of  $c$  annuitants/pensioners,  $\omega$  the extreme age and  $i_k^*$  the random interest rate earned in the period  $(k-1, k)$ , we have the two following different cases referred, respectively, to the first  $n$  years (i.e. *working period*) and to the subsequent years (i.e. the *retirement one*):

$$F(k) = F(k-1)(1+i_k^*) + N_k P \quad k = 1, 2, \dots, n-1,$$

$$F(k) = F(k-1)(1+i_k^*) - N_k R \quad k = n, n+1, \dots, \omega-x.$$

Here we report the portfolio fund value in three different forms, in dependence of the valuation time position:

$$F(h) = \sum_{i=0}^{h-1} N_i P \prod_{j=i+1}^h (1+i_j^*) + N_h P \quad h = 1, 2, \dots, n-1$$

$$F(n) = F(n-1)(1+i_n^*) - N_n R,$$

and

$$F(n+m) = F(n-1) \prod_{i=n}^{n+m} (1+i_i^*) - \sum_{p=n}^{n+m-1} N_p R \prod_{q=p+1}^{n+m} (1+i_q^*) + n N_{n+m} R,$$

this last with  $m = 1, 2, \dots, \omega - n$ .

The uncertainty risk measure, can be written as:  $UR = V[E[F(k)|T]]$ .

The application is based on a portfolio of 1000 policies issued on insureds aged 45, entering in the retirement phase at age 65, that is after 20 years. The annuity the insurer will pay in case of life of the pensioner is  $R=100$ , payable at the beginning of each year.

We will perform the risk analysis focusing on the retirement period, that is from age 65 on.

In Figure 4 we can observe the trend of the uncertainty risk assuming a fixed rate of 3% and the LC model for the premium calculation.

The index always increases, assuming very low values at the beginning of the retirement phase.

Differently from Figure 4, Figure 5 reports the Uncertainty Risk when the same hypotheses for the premium calculation and for the fund dynamics are considered. Specifically it means that also the premium is calculated by means of the same technical bases used for the fund dynamic. The UR moves differently from the preceding case, assuming generally definitely lower values and presenting a maximum around age 83, that is after 18 years of retirement. In particular, the uncertainty risk decreases after this age, showing a slightly increasing trend for very old ages.

**2.3. Application to the portfolio surplus.** In this section we analyze the uncertainty risk with regard to a portfolio of immediate identical policies, with benefits due in case of life to a cohort of  $c$  insureds aged  $x$  at issue (cf. Coppola et al., 2011).

We define the portfolio surplus at time  $t$  (cf. Lisenko et al., 2007; and Coppola et al., 2011) as the difference between the value of the assets accumulated until  $t$  and the value of the liabilities from  $t$  until the portfolio maturity.

Following the basic lines of Coppola et al. (2011), we introduce the liability  $V_t$  as follows:

$$V_t = \sum_{j>t} N_j Y_j v(t, j), \tag{5}$$

where  $v(t, j)$  is the value at time  $t$  of one monetary unit due at time  $j$ ,  $N_j$  the number of survivors at time  $j$ ,  $Y_j$  ( $j > t$ ) the difference at time  $j$  between the insurer's obligations and the net premiums.

Then we introduce the asset  $A_t$ :

$$A_t = \sum_{j \leq t} N_j X_j v(t, j),$$

where  $X_j$  is the difference between the premiums collected and the benefits due.

As clearly pointed out by Lisenko et al. (2007), the value of the assets, also called *retrospective gain* in actuarial context, is the accumulated value of premiums collected minus benefits paid before  $t$ ; the value of the liabilities, also called *prospective loss*, is the discounted value of benefits minus premiums due after  $t$ .

Now, considering the surplus  $S_t$  of a portfolio of identical policies issued to  $c$  insured aged  $x$ , it holds:

$$S_t = \sum_j N_j X_j v(t, j),$$

with  $X_j$  the difference between premiums and benefits and:

$$v(t, j) = (1+i(t, j))^{\text{sign}(t-j)},$$

where  $i(t, j)$  is the structured interest rate in the time interval between  $t$  and  $j$  and:

$$\text{sign}x = \begin{cases} \frac{|x|}{x}, & \text{for } x \neq 0 \\ 1, & \text{for } x = 0 \end{cases}.$$

Applying the uncertainty risk measure introduced in Section 1 to  $S_t$ , we obtain:

$$UR = V[E[S_t|T]] = V\left[\sum_j E[1_j] X_j E[v(t, j)|T]\right],$$

with  $1_j$  the indicator function which assumes the value 1 if the claim happens at time  $j$ , 0 otherwise.

Finally we obtain:

$$V[E[S_t|T]] = c^2 V\left[\sum_{j,j} p_x X_j E[v(t,j)|T]\right].$$

The surplus analysis provides information about the financial position of the business under consideration. It improves when the loading factor, say  $\theta$ , applied to premium calculation increases.

To this in-depth analysis the following observations are relevant (cf. Coppola et al., 2011):

**Proposition 2.** For each  $i \in (1, 2, \dots, n)$ , under the same hypotheses in section 1 and posing  $N_j$  multinomial  $(c, j, p_x)$ ,  $\frac{S_t}{c}$  converges in distribution to the random variable  $\Gamma_t = \sum_j X_{jj} p_x v(t, j)$ .

To this proposition we add the following result involving the standard deviation of the surplus (with respect to the systematic demographic uncertainty), considered per unit of mean surplus:

**Remark 1.**

$$\lim_{\theta \rightarrow \infty} \frac{\sqrt{V[E[S_t|T]]}}{E[S_t]} = \frac{\sqrt{V[\sum_{j,j} p_x X_j E[v(t,j)|T]]}}{\sum_{j,j} p_x X_j E[v(t,j)]}.$$

This is evident, because in the case of immediate annuity the only premium to capitalize is the single premium paid at the issue time.

The portfolio we consider in this application consists of  $c=1000$  immediate life annuities. In Figure 6 the Uncertainty Risk of the portfolio surplus is represented, for age at issue varying from 40 to 65 and time of valuation from 5 to 40 years.

In Figure 6 we can observe that the uncertainty risk index trend increases when the age at issue increases, pointing out the highest values 30 years after the issue time.

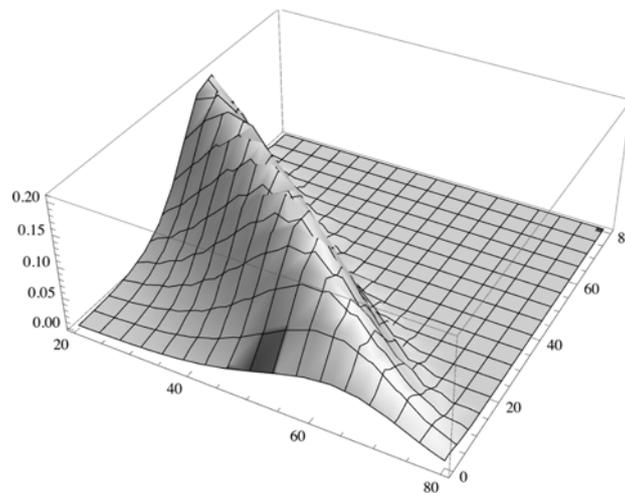


Fig. 1. The uncertainty risk of the average cost per policy, age from 20 to 80, time from 0 to 80

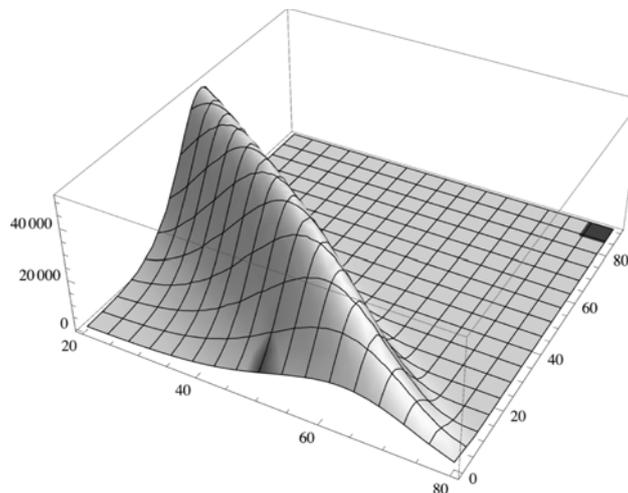


Fig. 2. The uncertainty risk of the initial value of the portfolio, age from 20 to 80, time from 0 to 80,  $c = 500$

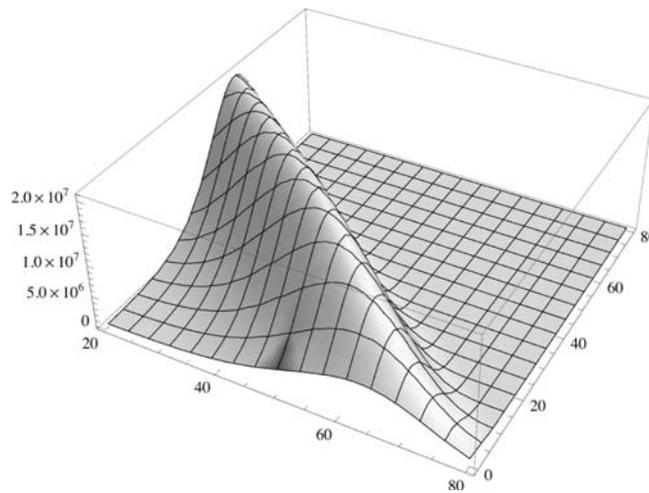


Fig. 3. The uncertainty risk of the initial value of the portfolio, age from 20 to 80, time from 0 to 80,  $c = 10000$

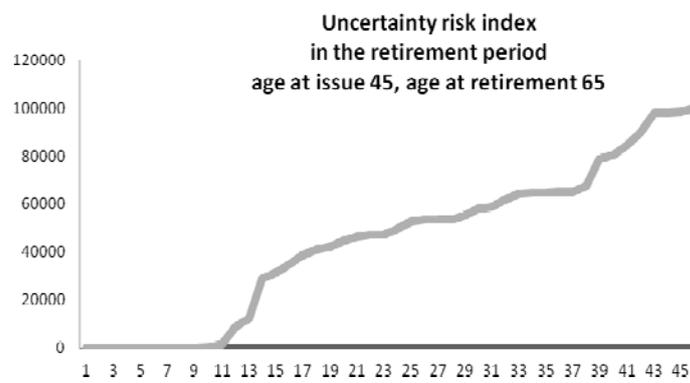


Fig. 4. The uncertainty risk trend in fund valuation. Different hypotheses for premium and fund



Fig. 5. The uncertainty risk trend in fund valuation. Same hypotheses for premium and fund

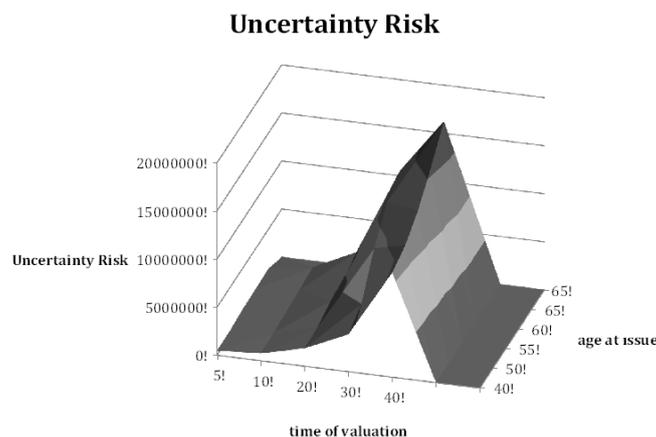


Fig. 6. The uncertainty risk index of the portfolio surplus,  $x = 40, 50, 55, 60, 65$ ;  $t = 5, 10, 20, 30, 40$

## Conclusions

To conclude, we would like to briefly illustrate the strategic significance of the results presented in this work. Our study engaged with the issue of demographic model selection for actuarial valuations within the picture of life insurance portfolios, drawing particular attention to the systemic impact of survival trend improvements. Within this context, knowledge of the risk linked to model selection, the uncertainty risk, is of prime importance, as it paves the way for the implementation of effective hedging strategy and management line. Moreover, formalizing and quantifying the uncertainty risk is useful within a predictive context, as well as within a comparative context: it can serve as a basis for business decision and can be easily updated year-by-year as the demographic and investment conditions modify.

By consequence, this work was especially concerned with the definition and quantification of uncertainty risk informed by the randomness of this particular choice; the analysis is performed taking into account also the interplay between the systemic demographic risk and the investment risk, whose impact is averaged out within the uncertainty risk valuation.

Large attention has been given to the graphic representation of risk index dynamics, too; empirical evidence shows, for example, a significant risk concentration in given age and time ranges. Being the evidences all got on financial quantities affected by both the return rate and the mortality randomness, the methodology allows for identification of precise hedging actions and adequate contractual structures.

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## Appendix

We briefly recall the main notations concerning the three survival models used in Section 2. See also Brian (2013), Blake et al. (2006), Lee et al. (1992), Booth et al. (2002) for a deeper understanding here.

### A1. The LC model

The Lee-Carter (LC) model involves the logarithm of the mortality rate at a given age  $x$  in a given year  $t$ , as follows:

$$\ln m_{x,t} = \alpha_x + \beta_x k_t + \varepsilon_{x,t}$$

with

- ◆  $m_{x,t}$  = death rate at the age  $x$  in the year  $t$ ;
- ◆  $\alpha_x$  = age-specific component (not dependent on time) at age  $x$ ;
- ◆  $\beta_x$  = age-specific component (not dependent on time), which incorporates mortality variations linked with variations of the general mortality level (sensitivity parameter);
- ◆  $k_t$  = time component which expresses the general mortality level in year  $t$ ;
- ◆  $\varepsilon_{x,t}$  = error term with zero mean and finite variance.

In particular,  $\alpha_x$  is the mean of  $\ln m_{x,t}$  throughout the observation period.

In order to estimate the parameters, which are not fully observable, Lee and Carter used the following normalizing positions:

$$\sum_t k_t = 0$$

$$\bar{\alpha}_x = \frac{\sum_t \ln m_{x,t}}{n} = \ln \left( \prod_t m_{x,t} \right)^{\frac{1}{n}}$$

$$\sum_x \beta_x = 1$$

$$\bar{k}_t = \sum_x \ln m_{x,t} - \sum_x \alpha_x$$

$\bar{\beta}_x$  can be obtained by a linear regression and finally:

$$\ln \bar{m}_{x,t} = \bar{\alpha}_x + \bar{\beta}_x$$

Lee and Carter modeled  $k_t$  by means of an ARIMA(0,1,0) process.

### A2. The BMS model

The Booth-Maindonald-Smith (BMS) model improves the LC model, by modifying the time component and studying the optimal fitting period, coherently with linearity of the time component.

In particular, the model is the following:

$$\ln m_{x,t} = \alpha_x + \sum_{i=1}^n \alpha_x^{(i)} k_x^{(i)} + \varepsilon_{x,t}$$

in which

- ◆  $m_{x,t}$  = death rate at the age  $x$  in the year  $t$ ;
- ◆  $\alpha_x$  = age-specific component (not dependent on time) at age  $x$ ;
- ◆  $\alpha_x^{(i)}$  = age-specific components (not dependent on time), which incorporate mortality variations as the general mortality level varies;
- ◆  $k_t^{(i)}$  = time component which expresses the general mortality level in year  $t$ ;
- ◆  $\varepsilon_{x,t}$  = error term;
- ◆  $n$  = the rank of the approximation;

### A3. The CBD model

The Cairns-Blake-Dowd (CBD) model involves a two factor model for describing the evolution in time of the mortality curve.

Specifically, the model focuses on the ratio of the mortality  $q_y$  to the survival rate  $p_y$ :

$$\ln \frac{q_y}{p_y} = A_1 \times A_2 y + error.$$

For  $A(t) = (A_1(t), A_2(t))$  Cairns, Blake and Dowd adopt the model

$$A(t+1) = A(t) + \mu + CZ(t+1),$$

with  $\mu$  is a constant  $2 \times 1$  vector,  $C$  is a constant  $2 \times 2$  upper triangular matrix,  $Z(t)$  is a two-dimensional standard normal random variable.