“Growth Optimal Investment Strategy Efficacy: An application on long run Australian equity data”

Growth Optimal Investment Strategy Efficacy: An application on long run Australian equity data

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Abstract

A number of investment strategies designed to maximise portfolio growth are tested on a long run Australian equity data set. The application of these growth optimal portfolio techniques produces impressive rates of growth, despite the fact that the assumptions of normality and stability that underlie the growth optimal model are shown to be inconsistent with the data.

Growth optimal portfolios are constructed by rebalancing the portfolio weights of 25 Australian listed companies each month with the aim of maximising portfolio growth. These portfolios are shown to produce growth rates that are up to twice those of the benchmark, equally weighted, minimum variance and 15% drift portfolios. The key to the success of the classic, no short-sales, growth optimal portfolio strategy lies in its ability to select for portfolio inclusion a small number of Australian stocks during their high growth periods.

The study introduces a variant of ridge regression to form the basis of one of the growth focused investment strategies. The ridge growth optimal technique overcomes the problem of numerically unstable portfolio weights that dogs the formation of short-sales allowed growth portfolios. For the short sales not allowed growth portfolio, the use of the ridge estimator produces increased asset diversification in the growth portfolio, while at the same time reducing the amount of portfolio adjustment required in rebalancing the growth portfolio from period to period.

Key words: Growth Optimal Portfolios, Australian Equity Returns, Feasible Investment Strategy, Ridge Regression.

Introduction

The expected rate of growth of value is considered by many investors to be the preeminent characteristic of an investment portfolio. Ways to construct portfolios that maximise expected growth are well documented[1]. Considering the importance of expected portfolio growth to both professional and retail investors, it is surprising that so few examples of studies that focus on the empirical strategies to maximise portfolio growth exist. This study aims to redress this deficiency by applying growth optimal techniques to long-run Australian equity data. As a first step in the study, let us set out a stochastic model of asset price evolution, upon which the growth optimising investment strategy will be based.

Suppose that investment choice is confined to *n* assets, each governed by geometric Brownian motion (generalised Wiener process). That is, the value of the i^{th} asset, V_i, evolves as[2]

$$dV_i(t) = \mu_i V_i(t)\,dt + V_i(t)\,dz_i\,, \tag{1}$$

where, μ_i is the i^{th} asset's rate of drift and *z* is a Wiener process with zero mean and variance, σ_i^2. The expected rate of growth of the asset, $E[G_i]$, over time *t*, can, using Ito's lemma, be derived as

$$E[G_i] = E\left[\ln\left(\frac{V_i(t)}{V_i(0)}\right)\right] = \left(\mu_i - \frac{1}{2}\sigma_i^2\right)t\,. \tag{2}$$

The variance of this growth is

$$\sigma_{G_i}^2 = \sigma_i^2\, t\,. \tag{3}$$

[1] See Hakansson (1971), Luenberger (1998) and Hunt (2002).
[2] The derivation of the portfolio dynamics follows Luenburger (1998, pp. 428-429).

Consider the dynamics of a portfolio constructed using specific asset weights, w_i. The rate change of portfolio value is thus the weighted sum of the rates of change of the individual assets:

$$\frac{dV_p}{Vp} = \sum_{i=1}^{n} w_i \frac{dV_i}{V_i} \cdot \tag{4}$$

Assuming that the n assets are correlated through the Wiener process, i.e., $\text{covariance}(\mathrm{dz_i}, \mathrm{dz_j}) = \sigma_{\mathrm{i,j}}\, dt$, the value of the portfolio, $V_p(t)$, also follows geometric Brownian motion with per period, expected growth, g_p, given by

$$\begin{aligned} g_p &= \frac{1}{t} E\left[\ln\left(\frac{V_p(t)}{V_p(0)}\right)\right] = \frac{1}{t}\left(\mu_p - \frac{1}{2}\sigma_p^2\right)t \\ &= \mathbf{w}^T\boldsymbol{\mu} - \frac{1}{2}\mathbf{w}^T\boldsymbol{\Omega}\,\mathbf{w}, \end{aligned} \tag{5}$$

where, $\boldsymbol{w}^T = (w_1, ..., w_n)$ is the vector of portfolio weights, $\boldsymbol{\mu}^T = (\mu_1, ..., \mu_n)$ is the vector of asset drift parameters and $\boldsymbol{\Omega}$ is a matrix containing n^2 variance and covariance terms, $\sigma_{i,j}$.

It is evident from (5) that the rate of growth of a portfolio of assets is governed by the choice of the individual asset weightings, $\boldsymbol{w}$. Naturally, the structure of $\boldsymbol{w}$ may be fashioned to maximise the expected rate of growth. The portfolio, w^*, that maximises expected portfolio growth is referred to as the *growth optimal portfolio*.

A strategy designed to maximise expected growth has an obvious and intuitive appeal. Moreover, maximising expected growth has strong theoretical support. Consider the broad class of power utility of wealth functions:

$$U(W) = \frac{1}{\gamma} W^{\gamma} \cdot \tag{6}$$

It is easily shown that for individuals possessing a utility function such as (5), the problem of maximisation of expected utility of wealth after n periods, W_n, reduces to the myopic strategy of the maximisation of wealth over one period, W_1[1]. Further, if γ is small, the expected value of power utility $E[U(W)]$ is closely approximated by

$$\begin{aligned} E[U(W_1)] &\approx E[\ln(W_1) + \frac{\gamma}{2}(\ln W_1)^2] \\ &\approx E[g] + \frac{\gamma}{2}(\sigma^2 + (E[g])^2]. \end{aligned} \tag{7}$$

Thus, when γ is small, it follows that the only two variables of interest in the quest to maximise expected utility of n period wealth, are expected growth rate and the variance of the growth rate. Investors with a log utility function $U(W_n) = \ln(W_n)$, which is the limit case of (7) when $\gamma \to 0$, will choose between investments based solely on expected portfolio growth[2].

Investment techniques based on optimising expected growth have appeal to both theorists and practitioners as they:

- are consistent with asset diversification,
- are consistent with n period utility maximisation,
- maximise expected terminal value of wealth, and
- minimise the expected time required for accumulated wealth to reach any specified threshold value.

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[1] See Luenberger (1998, pp. 425-427).

[2] Luenberger (1993) provides a broader rationale for basing portfolio choice on expected growth using so called tail preference theory.

The theoretical attractiveness of maximal growth portfolios is clear. What is less apparent is whether or not investment strategies based on growth portfolios are efficacious. The aim of this paper is to examine the suitability of growth optimal portfolio techniques to the Australian equity investment environment.

Data

Hakansson (1971) suggested that growth optimum portfolios dominate all other portfolios in the long run. While Merton and Samuleson (1974) pointed out the fallacy in this argument, it remains true that it is easier to identify the characteristics of alternative investment portfolios when observed over a long period of time. The desire to test the efficacy of growth-oriented investment led us to seek out a long run Australian equity data set. The study applies the growth optimal portfolio investment strategy techniques to 25 years of monthly data, starting in April 1977 and ending in March 2002. The data were obtained from Reuter's Australia's Beacon data service.

The data set comprises price observations on 25 Australian listed companies. These companies selected themselves, being the only corporations currently included in the ASX's 150 largest capitalised companies (as of March 2002) whose prices from March 1972 were recorded in the Beacon data tables[1]. The price data were transformed into measures of periodic growth (or returns) by using the formula for continuous compounding:

$$g_{i,t} = \ln(P_{i,t} / P_{i,t-1}), \qquad (8)$$

where P_t is the price of asset i in month t, and $g_{i,t}$ is the growth of asset i in month t[2].

Table 1 displays annualised statistics on rates of growth and volatility of growth for the companies included in the data set. The annual rate of growth of asset i, $\hat{g}_i$, was estimated as the sample aggregate growth divided by the 25 years of the sample. The estimate of the asset drift rate, $\hat{\mu}_{i,}$, was computed as $\hat{g}_i - \hat{\sigma}_i^2 / 2$, where $\hat{\sigma}_i^2$ is the estimate of the i^{th} asset variance.

As expected, there is some survivor effect to be evident in the 25-company data set. Table 1 shows that the All Ordinaries Index grew at a rate of 9.68% p.a. whereas the equally weighted portfolio grew at a rate of 11.71% p.a.

Table 1

Australian Equity Performance Summary Statistics

Code	Name	Growth (g, % p.a.)	Rank	Drift* (μ, % p.a.)	Rank	Volatility (σ, % p.a.)	Rank
1	2	3	4	5	6	7	8
AGL	Australian Gas Light	13.28%	7	17.99%	6	30.71%	8
AMC	Amcor	8.62%	19	11.22%	19	22.80%	23
ANZ	ANZ Bank	13.09%	8	16.11%	12	24.57%	19
BHP	BHP Billiton	12.61%	10	16.06%	13	26.25%	17
BIL	Brambles Industries	14.74%	5	17.75%	7	24.52%	20
CML	Colonial Mutual	11.49%	13	14.10%	16	22.82%	22
CSR	Colonial Sugar	3.95%	23	7.75%	22	27.55%	15
FGL	Fosters Brewing	11.10%	14	15.46%	14	29.53%	10
GMF	Goodman Fielder	3.72%	24	7.28%	23	26.71%	16

[1] Although the growth techniques were applied to 300 monthly observations, beginning in April 1977, the actual data set employed by the study starts in April 1972, to facilitate historical parameter estimation.

[2] The expected return over a very short period of time is $\mu \Delta t$. However, over a longer period of time the expected return is $\mu - \sigma^2/2$. As Hull (2000, pp. 240-241) notes, "the term *expected return* is ambiguous. It can either refer to μ or $\mu - \sigma^2/2$". When the term expected return (or symbol r) is used in this paper it is in reference to the drift term, μ, in a generalised Wiener process.

Table 1 (continuous)

1	2	3	4	5	6	7	8
GPT	General Property Trust	4.29%	22	5.57%	25	16.01%	25
LLC	Lend Lease	12.48%	11	16.30%	11	27.64%	14
MAY	Mayne Nicholas	9.04%	18	13.20%	17	28.83%	11
MIM	Mt. Isa Mining	-2.70%	25	5.73%	24	41.06%	2
NAB	National Australia Bank	12.67%	9	15.22%	15	22.60%	24
NCP	News Corporation	23.49%	2	32.86%	2	43.29%	1
ORI	Orica	6.05%	20	10.12%	20	28.51%	12
PDP	Pacific Dunlop	6.05%	21	10.04%	21	28.27%	13
QBE	QBE Insurance	18.60%	3	23.21%	3	30.39%	9
RIO	Rio Tinto	10.57%	15	16.79%	9	35.27%	7
SRP	Southcorp	14.35%	6	17.68%	8	25.79%	18
STO	Santos	15.33%	4	21.79%	4	35.94%	5
WBC	Westpac Bank	9.54%	17	12.41%	18	23.97%	21
WMC	Western Mining	9.86%	16	16.78%	10	37.21%	4
WPL	Woodside Petroleum	11.68%	12	18.00%	5	35.54%	6
WSF	Westfield Holdings	38.82%	1	47.23%	1	41.02%	3
ZAORD	All Ordinaries Index	9.68%		11.61%		19.65%	
Equal	Equally weighted portfolio	11.55%		13.35%		18.95%	

* The implied μ of equation (1) was computed for each stock by using equation (4) as the rate of growth plus the half the variance.

Some impressive individual growth performances are evidenced in Table 1. Westfield Holdings and News Corp have grown on average 39% p.a. and 23% p.a. respectively. At the other end of the performance spectrum lies MIM who managed an almost 3% p.a. decline in value over the 25 year period. Table 1 contains some superficial evidence of a positive relationship between historical share growth and the volatility of that growth. For example, the two highest growth stocks are also the two most volatile ones. Further analysis reveals that the correlation and the rank correlation between growth and volatility for the 25 stocks are 0.37 and 0.21 respectively.

Testing the Assumptions of the Growth Model

There are a number of assumptions implicit in the model of growth upon which the investment strategies tested in this paper are based. Most obviously, the Wiener process of equation (1) assumes investment returns are normally distributed. Possibly less obvious is the model's reliance on the stability of stochastic process parameters of μ and Ω. The degree to which these assumptions are consistent with the features of the historical data set ought to provide a guide to the likely success or otherwise of growth optimal investment strategies.

Normality

The 25 companies and the benchmark equally weighted portfolio and All Ordinaries Index series were tested for normality of returns with the results recorded in Table 2. Three tests of normality, based on skewness and kurtosis measures, were applied to the data set. The results of these tests reveal that the period-by-period returns in the data set were far from normally distributed.

22 out of the 25 stocks displayed significant skewness. In addition, all 25 stocks had returns that were significantly leptokurtic (at the 1% level). Naturally, the Jacque-Berra statistics, which jointly tests for skewness and kurtosis, rejected normality in all cases, including the two benchmark series.

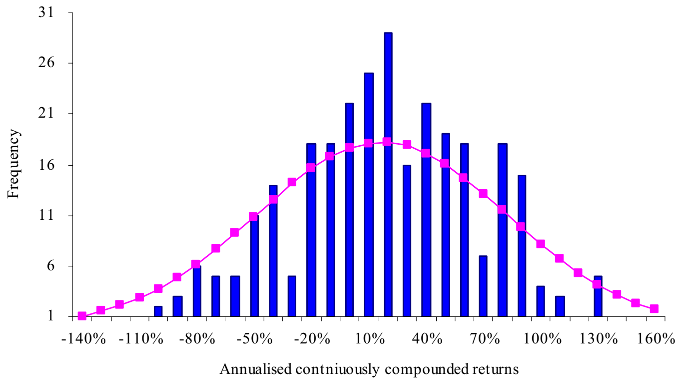

Fig. 1. Distribution of Returns of an Equally Weighted, 25 Stock Portfolio

Figure 1 depicts the distribution of returns for the equally weighted portfolio compared to its equivalent normal distribution. The typical peaked centre of a leptokurtic distribution is clearly displayed.

Table 2

Tests of Normality[1] and Stability

Code	Skewness		Kurtosis†		Jacque-Berra[1]		ANOVA		Kruksal-Wallis		Variance Ratio	
1	2		3		4		5		6		7	
AGL	-0.66	**	7.02	**	109.36	**	1.06		2.08		4.72	**
AMC	-0.58	**	4.76	**	76.30	**	0.37		1.82		2.58	*
ANZ	-0.54	**	2.05	**	40.23	**	0.34		0.80		1.94	
BHP	-0.19		1.51	**	20.73	**	0.96		3.02		2.27	
BIL	-0.91	**	4.50	**	98.03	**	0.66		3.74		2.02	
CML	-0.98	**	7.19	**	137.96	**	0.64		4.38		2.74	*
CSR	-0.53	**	2.74	**	48.22	**	0.06		0.89		3.56	**
FGL	-1.73	**	19.40	**	391.97	**	1.66		5.86		5.38	**
GMF	-0.64	**	4.41	**	75.65	**	2.49	*	7.17		3.92	**
GPT	-0.77	**	4.12	**	81.16	**	0.87		4.22		2.32	
LLC	-1.62	**	11.51	**	275.80	**	1.23		2.93		4.54	**
MAY	-0.95	**	6.27	**	123.83	**	0.60		3.25		2.21	
MIM	-0.66	**	3.82	**	69.78	**	0.04		0.89		4.00	**
NAB	-0.70	**	2.76	**	58.95	**	0.32		1.29		2.49	*
NCP	-1.39	**	9.84	**	219.03	**	2.67	*	9.08		6.58	**
ORI	-0.41	**	5.41	**	76.19	**	0.78		3.66		2.37	
PDP	-0.64	**	3.34	**	61.96	**	3.06	*	12.59	*	1.87	
QBE	-2.14	**	13.49	**	398.45	**	1.00		4.30		3.62	**
RIO	-1.42	**	13.66	**	271.06	**	0.42		3.44		8.52	**

[1] The skewness and kurtosis tests are based on the following. For a normal distributed random variable, x, the skewness coefficient, $\theta_1 = E[(x-\mu)^3]/\sigma^3$ estimated from a sample of size n is distributed as $\hat{\theta}_1 \approx N(0, 6/n)$. The coefficient of kurtosis, $\theta_2 = E[(x-\mu)^4]/\sigma^4$ is distributed as $\hat{\theta}_2 \approx N(3, 24/n)$, where E is the expectation operator, μ is the mean and σ is the standard deviation. The Jacque-Berra statistics, J, where: $J = n\left(\frac{\hat{\theta}_1^2}{6} + \frac{(\hat{\theta}_2 - 3)^3}{24}\right) \approx X^2(2)$

Table 2 (continuous)

1	2		3		4		5		6		7	
SRP	-0.62	**	7.23	**	109.70	**	0.75		3.65		2.19	
STO	0.18		3.39	**	43.97	**	3.16	*	9.50	*	7.01	**
WBC	-0.50	**	3.64	**	57.93	**	0.32		1.77		2.75	*
WMC	-0.36	*	4.25	**	59.39	**	0.16		0.31		3.61	**
WPL	0.15		1.97	**	25.72	**	0.37		1.20		6.68	**
WSF	5.29	**	67.36	**	2241.86	**	0.14		5.64		26.28	**
ZAORD	-3.27	**	31.04	**	921.65	**	1.29		5.54		5.18	**
Equal	-2.80	**	25.30	**	709.05	**	0.98		5.30		5.20	**

* indicates significance at the 5% level, ** indicates significance at the 1% level.

† The kurtosis figure displayed was computed using Excel's KURT() function and is equal to the traditional measure of kurtosis less 3.

The results of the analysis of skewness and kurtosis allow us to confidently conclude that the data upon which we are to test the growth optimal portfolio strategies are non-normal. Exactly how the non-normality will impinge upon the investment results is problematical. For example, it is not clear that excessive kurtosis will have a deleterious effect on the growth optimal investment strategies. Of more concern is the question as to the stability of the distributional statistics overtime.

Serial Stability

The expected growth rate for each stock, the variance of that growth rate and the covariances between each stock's growth rates are essential inputs to the process of determining growth optimal portfolio weights. Thus any serial instability in these input parameters will imperil the success of any investment strategy based on an assumption of parameter constancy. The growth optimal strategy relies on the stability of the input parameter estimates.

Table 3

Sub-period Statistics

Period	All Ordinaries Index		Equally weighted portfolio	
	Growth	Volatility	Growth	Volatility
1977-82	8.70%	19.79%	13.25%	18.45%
1982-87	12.48%	19.76%	14.06%	18.04%
1987-92	-1.31%	29.38%	4.50%	27.76%
1992-97	8.51%	12.91%	8.59%	12.32%
1997-02	7.55%	12.93%	6.61%	13.42%

Table 3 sets out estimates of the average growth and volatility of growth, for the two benchmark series, for the five equal sub-periods that make up the overall data set period. Casual analysis of the range of sub-period estimates suggests parameter instability. However, the result of applying formal tests for instability of the mean of growth for the individual stocks does not lead to the conclusion that these are unstable. Analysis of variances indicates instability in only 4 out of the 25 stocks. The Kruksal-Wallis test, which is the more suitable test given the non normality of the data, rejects the hypothesis of constancy of growth rates in all five sub-periods for only 2 out of the 25 stocks.

The proposition that variance of growth rates is identical in each of the 5 sub-periods, was checked by using Hartley's test for homogeneity of variance. The ratio of the largest sub-period variance to the smallest sub-period variance, which is the key statistics in Hartley's test, is displayed in Table 2[1]. Hartley's test indicates that the presence of serial instability of variance of growth rates exists in many of the sample stocks. The null hypothesis of equality of sub-period was rejected, at the 5% level at least, for 17out of the 25 companies.

The variance for each stock is an input into the formation of growth optimal portfolios. However, it is the full covariance matrix that is the essential input item and the variances represent only a small proportion of the larger covariance matrix[2]. However, the preceding evidence of variance instability justified further research to ascertain whether the variance instability was also mirrored in covariance instability.

A test of the hypothesis of equality of sub-period covariance matrices employs the Box's M statistic, where

$$M = n\sum_{i=1}^{m} \ln|\mathbf{\Omega}| - n_s \sum_{i=1}^{m} \ln|\mathbf{\Omega}_i|, \tag{8}$$

where m is the number of sub-periods, $n = m\ n_s$ is the number of observations in the full sample, n_s is the number of observations in each sub-period, $|\mathbf{\Omega}|$ is the determinant of the overall, p dimensioned, covariance matrices and $|\mathbf{\Omega}_i|$ is the determinant of the i^{th} sub-period covariance matrix. Pearson (1969) shows that for large p, M is distributed as $b\ F_{f1,f2}$ [3].

The sample M/b was computed as 1.75. This is to be compared to the 1% critical F of 1.09. Hence, it must be concluded that the sample data covariance matrix is not stationary.

The preceding results do not allow much scope for optimism as to the successful application of investment techniques based on growth optimal portfolios, as the techniques rely on an assumption of normality of period-by-period growth rates, and an implicit assumption of the stability of the distributional parameters contained within the expected growth rates and the covariance matrix of growth rates. Contrary to these assumptions, the analysis has shown that long run Australian equity data are leptokurtic and somewhat skewed and are characterised by a non-stationary covariance matrix. However, despite the facts of the situation, we proceeded to test efficacy of growth optimal portfolio investment techniques using the historical Australian data.

Application of Growth Optimal Portfolio Investment Techniques

This paper attempts to test a simple, practical investment strategy based on portfolios selected to have maximum expected growth rate. Testing any proposed investment strategy on the historical data involved stepping through each of the 300 monthly observations on the return of 25 Australian companies. At any period, k, the following steps are undertaken:

1. The data on the previous n periods are employed to provide estimates of the expected growth rate for each stock in the sample and to estimate each element of the 25 x 25 growth rate covariance matrix.
2. The expected growth and covariance matrix estimates are used to produce growth optimal portfolio weights, w_k.
3. The return on this portfolio in the next, i.e. k+1, period is computed.
4. The time-frame is moved forward one observation.

Steps 1 to 4 are repeated until the data set is exhausted.

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[1] The ratio of the highest to the lowest sub-period variance is theoretically distributed as $F_{n1,n2}$, where $n1$ is the number of sub-samples and $n2$ is the number of observations in each sub-sample, *ie* $F_{5,60}$ in our case. For more fulsome explanation of Hartley's test see Berenson and Levine (1992, pp. 506-507).

[2] A covariance matrix contains n variance terms and $n\ (n-1)$ covariance terms. n = 25 in this study. Thus, for this study the variances represent only 25/600=4% of the terms in the covariance matrix.

[3] The calculation of M/b is rather daunting with $f1$, $f2$ and b having more terms than a Mettalica tour contract. See Pearson (1969, p. 219). An example of the use of M to test equality of covariance matrices can be found in Morrison (1976, pp. 252-253) (note, however, the error in Morrison's equation (2)).

Short-sales Allowed Portfolios

Growth optimal portfolios lie on a minimum variance frontier formed when portfolio variance is minimised for a range of expected portfolio drift (Figure 2). The short-sales allowed, growth optimal portfolio, $\boldsymbol{w}^*$, vector has the following structure[1]:

$$w^* = A\boldsymbol{\mu} + b, \tag{9}$$

where:

$$A = \Omega^{-1}\left(I - \frac{\iota\iota^T\Omega^{-1}}{\iota^T\Omega^{-1}\iota}\right),\ b = \frac{\Omega^{-1}\iota}{\iota^T\Omega^{-1}\iota}$$ and $\boldsymbol{\iota}$ is the unit vector.

It is our aim to proceed through the historical data set, estimating $\boldsymbol{\mu}$ and $\boldsymbol{\Omega}$, using these estimates to calculate the growth optimal weights, $\boldsymbol{w}^*$, and to use these weights to produce a set of one-step-ahead returns for each of the 300 observations in the data set. The success or failure of the growth optimal investment techniques will be judged on the nature of the one-step-ahead returns produced by the strategy. The returns on three alternative investment strategies will provide a base against which to measure the growth optimal techniques.

These benchmark portfolios are as follows:

1) the equally weighted portfolio,
2) the minimum variance portfolio, and
3) the portfolio with an expected drift of 15% p.a.

The equally weighted portfolio is a simple passive investment strategy and represents the absolute minimum "bar" against which alternatives ought to be measured. The minimum variance point (MVP) strategy aims to minimise portfolio variance regardless of the expected level of portfolio drift. Weights for the minimum variance portfolio (MVP), w_{MVP} are given by:

$$w_{MVP} = b = \frac{\Omega^{-1}\iota}{\iota^T\Omega^{-1}\iota}. \tag{10}$$

The final benchmark portfolio is one with an expected drift rate of 15% pa. The figure of 15%, while being arbitrary, is consistent with the historical record and is in general accord with Australian investors' expectations of reasonable share market returns.

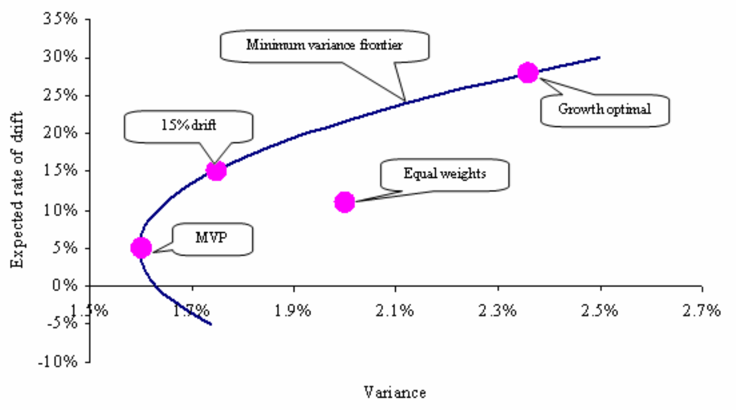

Fig. 2. Growth Portfolio and the Benchmark Portfolios

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[1] The structure of short-sales allowed growth optimal portfolios is extensively explored in Hunt (2002).

A stylised representation of the relative positions of the growth portfolio and the three benchmark portfolios is depicted in Figure 2[1].

The inverse of the covariance matrix, $\boldsymbol{\Omega}^{1}$, is necessary for the determination of the weights of the growth optimal portfolio, the MVP and the 15% drift portfolio. Unfortunately, a problem arises in the computation of $\boldsymbol{\Omega}^{1}$ due to the multi-collinear nature of the periodic stock growth rates. The empirical estimate of the 25-stock covariance, $\boldsymbol{\Omega}$, is at times close to being singular. The near singularity of $\boldsymbol{\Omega}$ results in a loss of numerical precision which in turn results in estimates of individual stock weights, w^*_i, that gyrate wildly from observation to observation.

The replacement of $\boldsymbol{\Omega}$ with an amended covariance matrix, $\boldsymbol{\Omega}_+$ in the estimation process provides a solution to the multi-collinearity problem, where:

$$\boldsymbol{\Omega}_+ = \boldsymbol{\Omega} + d\,I\,, \tag{11}$$

where d is a scalar and I is the identity matrix.

The approach embodied in (11) is analogous to the ridge solution to multi-colinearity in regression analysis[2]. The use of a non-zero d in (11) produces "biased" estimates of the growth optimal portfolio and the benchmark portfolios. As the pivot d increases, the ridge estimate of growth optimal portfolio weights, w_+^*, is biased away from the classic growth optimal portfolio weights towards the equally weighted portfolio. That is, in the limit:

$$\lim_{d \to \infty} w_+^* = \frac{1}{n}, \tag{12}$$

where n, the number of assets in the set, is 25 in our case. In other words, the ridge estimator produces weights that are a combination of the classic estimator weights and those of the equally weighted portfolio.

A decision to use a ridge estimator necessarily requires a particular value for d. A common approach in ridge regression analysis is to choose a value for d that provides "stabilised" estimates of the system parameters. We have taken a similar approach in choosing a suitable d on the basis of its influence on portfolio length defined by[3]:

$$length = \sqrt{(n\,\mathrm{w}^T \mathrm{w})}\,. \tag{13}$$

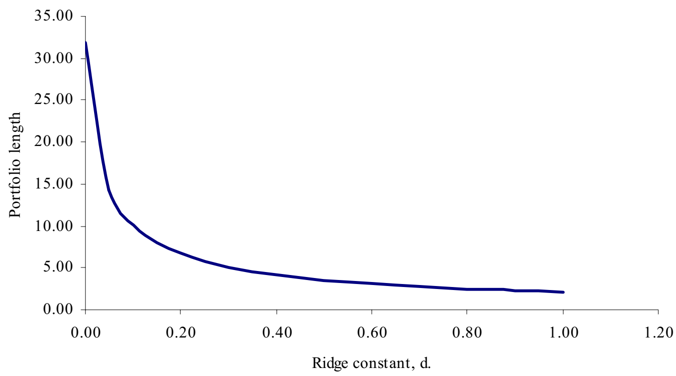

Fig. 3. Portfolio Length Versus the Size of the Ridge Constant

1 Hunt (2002) shows that the growth optimal portfolio lies on the minimum variance frontier drawn in expected drift-variance space.

2 Judge (1985, pp. 474-486) provides an exhaustive review of ridge estimators. An alternative form of ridge estimator, $\Omega_+ = \Omega + d\,D$ where d is a diagonal matrix of individual stock variances, was also considered. The weights produced by this ridge estimator are inversely proportional to the stock variances in the limit as $d \to \infty$.

3 Under this definition the minimum length portfolio, i.e. the equally weighted portfolio, has unit length.

Figure 3 plots the growth portfolio length (estimated over the entire sample) against *d*. It was decided, on the basis of this plot, that setting *d* equal to 0.35 (35%) represented a reasonable compromise between portfolio length and the portfolio weight bias[1].

The results of applying the MVP, the 15% drift and the growth optimal strategy, with the ridge constant set to both zero and 0.35, for parameter estimation period lengths of 3, 4 and 5 years, are set out in Table 4. These results need to be measured against the equally weighted portfolio, which is shown in Table 1 to have an average growth of 11.55% p.a. (and thus an aggregate growth of 288.7%) with a volatility of 18.9% p.a.

Table 4

Short-sales Allowed Portfolio Strategy Returns

	Ridge constant = 0.00			Ridge constant = 0.35		
	MVP	15% drift	Growth	MVP	15% drift	Growth
Estimation period = 3 years						
Aggregate	213.8%	282.0%	12018.2%	292.2%	284.1%	359.0%
Average	8.6%	11.3%	480.7%	11.7%	11.4%	14.4%
Volatility	25.0%	23.7%	2549.9%	18.7%	17.7%	22.4%
Estimation period = 4 years						
Aggregate	195.2%	225.9%	12420.5%	290.1%	281.0%	344.4%
Average	7.8%	9.0%	496.8%	11.6%	11.2%	13.8%
Volatility	20.4%	20.4%	1057.2%	18.7%	18.2%	21.8%
Estimation period = 5 years						
Aggregate	169.0%	254.9%	11305.1%	289.1%	291.4%	337.9%
Average	6.8%	10.2%	452.2%	11.6%	11.7%	13.5%
Volatility	19.8%	20.0%	699.9%	18.7%	18.1%	21.5%

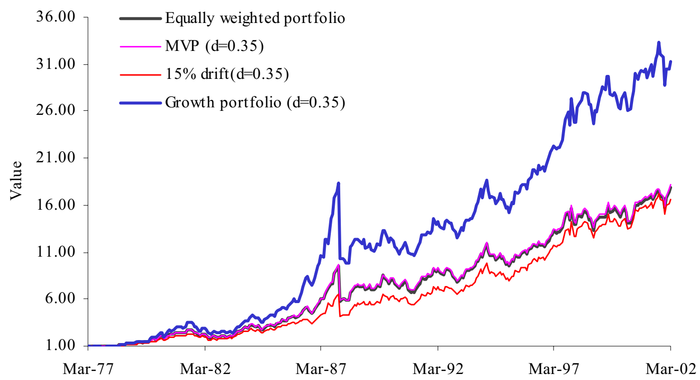

Fig. 4. Short -sales Allowed Portfolio Growth in Value

The first notable result is the volatility associated with the non-ridge (i.e. d=0.00) growth portfolios[2]. While the results in Table 4 show that the non-ridge portfolios were numerically attainable, they do not provide evidence that a growth oriented, short-sales allowed, investment

[1] The growth portfolio estimated with *d*=0.35 has a length that is less than 2% of that of the classic, *d*=0.0, growth portfolio.
[2] With 25 assets in the portfolio, the maximum rank of the covariance matrices is *n*-26, where *n* is the number of months in the estimation period. The portfolio weights estimated over the 3-year period are thus particularly susceptible to the problem of multicollinearity.

strategy was realistic or feasible. The short-sales allowed, zero ridge constant, growth portfolios had gearing ratios that any investor would find impractically high. For example, the average length of the 4-year estimation period, zero ridge constant, growth portfolio exceeded a value of 300. The strategy routinely required an asset to be short-sold more than 1000%. The highly geared portfolios were naturally characterised by high volatility of investment returns.

The short-sales allowed growth portfolios, whose weights were estimated with a ridge factor of 0.35, were much better behaved than their zero ridge factor counterparts. The growth strategy portfolios outperformed the other bench marks by more or less than 3% p.a. depending on the length of the estimation period. It is worth noting similarity in performance of each of the three benchmark portfolios for d=0.35. The equally weighted portfolio, the MVP portfolio and the 15% drift portfolio each produced a rate of growth of a little under 12% p.a., with an associated volatility of about 18% regardless of the length of the estimation period[1]. In fact the performance of all four strategies, including the growth strategy, appears to be relatively independent of the length of the estimation period for both the classic, and the ridge non-ridge portfolios

Figure 4 shows the dollar extent of the superior performance by the short-sales allowed, ridge constant=0.35, 4-year estimation period, portfolio over the 25 years of the data set. As previously stated, the presence of short-sold shares in the portfolios of either professional or retail investors is not typical. An analysis of the results of growth portfolios where short-selling is not allowed will provide a more practical test of the growth investment strategy.

Short-selling not allowed growth portfolios

While the short-selling of stock in most equity markets, including Australian, is allowed, it is not typical. Trialling growth optimal portfolios where a no short-sales restriction is imposed on portfolio weights, is a more realistic test of the strategy. The results from testing no short-sales growth portfolios are set out in Table 5.

The no short-sales growth portfolio performances are impressive. The statistics recorded in Table 5 shows that the classic no short-sales, non-ridge (i.e. d=0), estimator produces portfolio growth rates in excess of 20% p.a. and up to 31% p.a., depending on the length of the input estimation period.

Table 5

No Short-sales Allowed Portfolio Strategy Returns

	Ridge constant = 0.00			**Ridge constant = 0.35**		
	MVP	15% drift	Growth	MVP	15% drift	Growth
Estimation period = 3 years						
Aggregate	260.6%	289.2%	792.5%	292.2%	295.7%	358.1%
Average	10.4%	11.6%	31.7%	11.7%	11.8%	14.3%
Volatility	16.2%	15.4%	45.7%	18.7%	17.8%	22.2%
Estimation period = 4 years						
Aggregate	258.7%	278.3%	563.0%	290.1%	284.8%	345.2%
Average	10.3%	11.1%	22.5%	11.6%	11.4%	13.8%
Volatility	16.5%	16.3%	37.7%	18.7%	18.5%	21.8%
Estimation period = 5 years						
Aggregate	262.8%	276.8%	569.0%	289.1%	264.6%	337.8%
Average	10.5%	11.1%	22.8%	11.6%	10.6%	13.5%
Volatility	16.7%	18.1%	36.2%	18.7%	19.0%	21.5%

[1] The equally weighted portfolio performance is of course independent of the estimation period.

The aggregate growth for the 2-year, 3-year and 4-year estimation period growth portfolios is depicted in Figure 5[1]. The extent to which the growth portfolios outpaced the benchmark portfolios is clearly evident. The growth portfolios' performance is even more impressive when stated in dollar terms. One dollar invested in the no short-sales, 3-year, 3-year and 5-year estimation period, growth optimal portfolio strategy in March 1977 would have returned $2,764, $278 and $295 respectively at the end of March 2002. These figures grossly exceed the return on the All Ordinaries Index and the equally weighted portfolio, of $11.24 and $18.61 respectively.

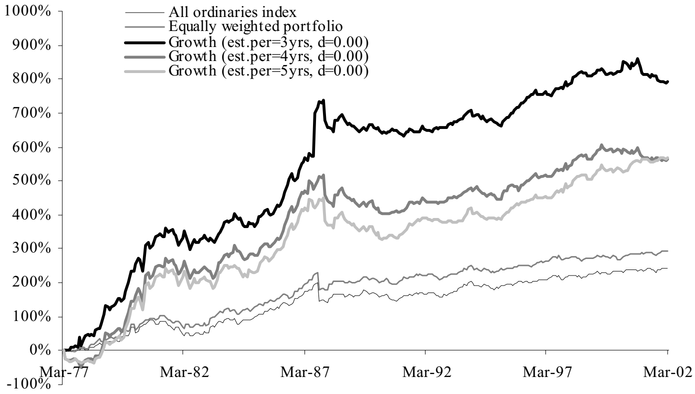

Fig. 5. Short-sales Not allowed Aggregate Growth Rates

The impressive performance of the no short-sales growth strategy begs further analysis. A couple of points about the performance of the growth portfolios can be made with reference to Figure 5. First the growth strategy accumulation of value is slower in the second half of the period than it is in the first half. Moreover, the performance in the last couple of years has been notably negative[2].

Table 6 sets out growth and volatility statistics for the classic and ridge short-sales not permitted portfolios and for each of the benchmarks, for the 4-year estimation periods. It is clear from Table 6 that the growth oriented strategies, while having considerably higher growth rates than the benchmark strategies, also have much higher volatility than the benchmark MVP and 15% growth and equally weighted strategies. The direct relationship between growth and volatility is also evident in the 3-year and 5-year estimation period as Table 5 shows. The evidence shows that no single investment strategy clearly dominates any other strategy. Indeed, the results of this study provide strong support for what Luenberger (1998) calls the log mean-variance model[3].

Low growth portfolios are associated with low volatility and high growth portfolios are associated with high volatility. The point is, however, that regardless of the cost in terms of volatility, the portfolios designed for maximal growth did produce quite remarkable rates of growth. It is worth investigating the source of this growth.

It is insightful to examine the average portfolio length and average number of included assets in the no short-sales growth optimal portfolios. The length of a no short-sales allowed portfolio is inversely indicative of its "diversity"[4]. The length of a no short-sales, 25-stock portfolio, can take values between one and five. The maximum portfolio length of 5 is achieved, for any 25-stock portfolio, when 100% of portfolio value is held in a single stock. At the other end of the spectrum is the maximally diversified, equally weighted portfolio with unit length.

[1] From here on, the analysis of results is restricted, for the sake of brevity, to the 2-year and the 3-year estimation period. The choice of these two estimation periods is justified as they yield both the highest and lowest growth rates respectively.

[2] Perhaps the poor recent performance of the growth portfolios may be summarised in the old adage, "if one lives by the sword (in this case an instrument finely crafted by Messes Murdoch and Lowey) one dies by the sword".

[3] Luenburger (1998, pp. 425-427).

[4] The use of vector length to measure diversity is related to Fernholz's measure of diversity $D_p=(\Sigma w_i^p)^{1/p}$. See Fernholz, Garvy and Hannon (1998).

Table 6

No Short-sales, 4-year Estimation Period, Portfolio Properties

Input estimation period	Portfolio type	Average growth rate	Volatility of growth rate	Average portfolio length	Average no. of included assets	Average turnover of assets
	Equal weights	11.55%	18.94%	1.00	25.00	0.00%
Classic no short-sales growth portfolios (d=0.00)	MVP	10.35%	16.49%	2.60	8.53	6.94%
	15% drift	11.13%	16.32%	2.39	8.56	11.33%
	Growth	22.52%	37.65%	4.36	1.71	11.83%
Ridge no short-sales growth portfolios (d=0.35)	MVP	11.61%	18.70%	1.00	25.00	0.25%
	15% drift	11.39%	18.53%	1.19	22.46	7.40%
	Growth	13.81%	21.75%	1.26	23.18	4.97%

Table 6 shows that the "classic" no short-sales allowed, growth optimal portfolio with an average length of 4.36, is at the lower end of the diversity spectrum. Moreover, Table 6 shows that this portfolio contains on average only 1.71 assets in each period. Further, Figure 7, which shows the distribution of the number of assets held in each period, reveals that for the majority of the 300 monthly periods, the no short-sales growth optimal portfolio consisted of a single asset.

Table 7

Growth Portfolio Included Companies*

Company	No. of appear-ances	First appears	Last appears	Growth while included (% p.a.)	Overall growth (% p.a.)
AGL	4	Jul-92	Dec-98		13.28%
AMC	4	Jul-92	Aug-92		8.62%
ANZ	1	Nov-92			
BHP	10	Dec-91	Mar-02		12.61%
BIL	18	Mar-90	Dec-99	-0.35%	14.74%
CSR	1	Mar-02			3.95%
FGL	11	Sep-82	Sep-83		11.10%
MAY	4	Jun-78	Jul-90		9.04%
MIM	1	Mar-02			
NAB	1	Jun-99			
NCP	118	Apr-77	Feb-02	18.05%	23.49%
ORI	3	Dec-94	Jan-95		
QBE	39	Jan-83	Sep-01	0.33%	18.60%
RIO	9	Apr-90	Sep-01		10.57%
SRP	2	Feb-84	Aug-91		14.35%
STO	74	May-77	Jun-83	51.27%	15.33%
WMC	23	Dec-88	Jan-02	0.95%	9.86%
WPL	26	Feb-89	Sep-01	-4.18%	11.68%
WSF	155	Feb-79	Mar-02	23.44%	38.82%

The statistics in the table is for stocks included in a no short-sales, classic growth optimal portfolio strategy, employing a 3 year estimation period.

The low number of assets held in each period results in an overall low rate of inclusion of individual companies over the 25 years. Table 7 sets out statistics relating to included stocks in a growth optimal strategy over 25 years. Only seven out of the 25 stocks are included in the strategy for 12 months or more. Table 7 shows that the 25-year strategy is dominated by three stocks: STO, NCP and WSF. NCP and WSF are Australian share market stellar performers producing fairly steady growth over 25 years of 23.49% p.a. and 38.82% respectively. In contrast, STO was a patchy performer. However, STO was well-performed stock over the period of its inclusion in the growth portfolio in the first five years of the 25-year trial period. STO contributed to the strategy 55% p.a. while it was included, compared to a more modest 15% p.a. over the whole period.

The success of the no short-sales allowed, growth optimal strategy appears to lie in its ability to identify companies during their periods of high growth for portfolio inclusion.

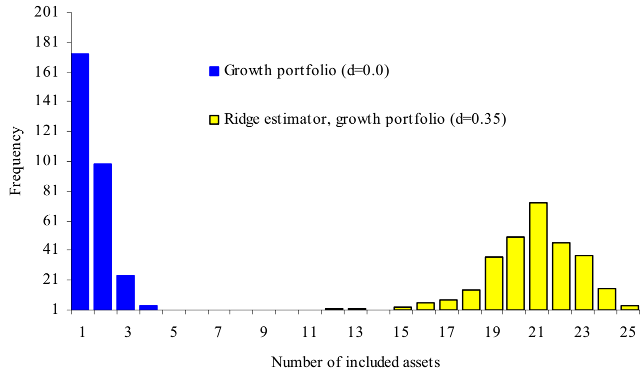

Fig. 7. Distribution of the Number of Assets Held in Growth Portfolios

Increased transaction costs are a practical consideration for any strategy that results in the placement of portfolio value in a few assets. Transaction costs will be significant if a strategy requires the flip-flopping of large asset weights from one asset to another. One can see that this is the case to a degree with the no short-sales allowed growth portfolios. Table 6 shows that the maintenance of this growth optimal portfolio strategy would have required on average a turnover of stock of about 12% per month. The transaction costs associated with any strategy that turns over 12% of a portfolio per month are considerable, and will significantly lower the effective rate of portfolio growth[1].

The ridge estimator was employed to produce short-sales allowed portfolio weights that were numerically stable and produced acceptable gearing levels. The justification for this use of a ridge estimator does not have the same force for short-sales not allowed portfolios. The no short-sales restriction considerably reduces the dimension of the covariance matrix that is inverted to produce portfolio weights. Classic short-sales not allowed portfolio weights are numerically stable and are by definition not geared. There is, however, an argument for the use of a ridge growth estimator in the short-sales not allowed context, on the grounds that it produces more diversified, less risky portfolios.

The ridge growth portfolio estimator is a combination of the classic growth optimal portfolio estimator and the maximally diversified, equally weighted portfolio. The size of the ridge constant, *d*, determines the extent to which a ridge growth estimator is biased away from the classic estimator towards the equally weighted estimator.

We have computed no short-sales, ridge, growth optimal portfolios using a ridge constant of 0.35 to facilitate comparison with the short-sales allowed results. Predictably, Table 6 reveals that the no short-sales allowed, ridge, growth portfolio performance lies somewhere between the performance of the classic growth optimal portfolio and the equally weighted portfolio. The no short-sales allowed, ridge, growth portfolio contains more assets (see Figure 6), is less risky and has a lower growth rate than the classic growth portfolio.

Conclusion

Growth optimal portfolio investment strategies were applied to a 25-year data set of 25 Australian companies. Initial statistical investigation of data provided no reason to be optimistic about the successful application of the growth techniques. The growth optimal technique assumptions of nor-

[1] For example, the cost of portfolio adjustment would be 2.9% p.a. if a round transaction cost was 2% of traded value.

mality and stability were violated by the nature of the Australian data. Returns on the 25 stocks were found to be skewed and leptokurtic and to have time varying variances and covariances. However, the growth optimal techniques perform well, despite the assumptions not being met.

The growth optimal portfolios, both short-sales allowed and short-sales not allowed, produced rates of growth that exceeded those of the benchmark portfolios. The classic no short-sales allowed, growth optimal portfolios produced impressive rates of growth that were more than double those of the benchmark portfolios. Analysis of the structure of these portfolios showed that, at any point in time, they consisted of a very small number of included stocks. The secret of the success of these portfolios appears to lie in their ability to select a few stocks during their high growth periods.

This study details the successful inclusion of a variant of ridge regression as the basis of a growth optimal strategy. The ridge growth optimal technique facilitated production of numerically stable weights for short-sales allowed portfolios. When short-sales were not allowed, the use of the ridge estimator produced more diversified growth portfolios.

There are two possible answers to the question of why the growth optimal techniques performed well in the face of non-normality and instability in the data. The first reason, which cannot be dismissed, is that the techniques work well on this particular data set by pure chance alone. The second explanation is that the assumptions of normality and stability are not necessary to the success of the technique. While the model used in this paper assumes normality in the Ito process, it may be that growth investment strategy is equally efficacious under alternative stochastic processes that allow kurtosis. Why does the investment strategy cope with distributional instability? Perhaps the use of a moving window estimation process may counter the problems arising from the instability of mean growth rates and growth rate covariances.

The study details the successful application of growth optimal techniques. There is, however, no evidence of the general superiority of growth optimal techniques. Growth portfolio strategies are also high volatility strategies. While the results of an empirical study such as this are necessarily limited to the specific market and to the specific time-frame of the study, the point that this study makes is, however, that regardless of their other properties and potential drawbacks, the portfolios designed for maximal growth did in fact produce quite remarkable rates of growth.

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