“No-arbitrage one-factor term structure models in zero- or negative-lower-bound environments”

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NO-ARBITRAGE ONE-FACTOR TERM STRUCTURE MODELS IN ZERO- OR NEGATIVE-LOWER-BOUND ENVIRONMENTS

Abstract

One-factor no-arbitrage term structure models where the instantaneous interest rate follows either the process proposed by Vasicek (1977) or by Cox, Ingersoll, and Ross (1985), commonly known as CIR, are parsimonious and analytically tractable. Models based on the original CIR process have the important characteristic of allowing for a time-varying conditional interest rate volatility but are undefined in negative interest rate environments. A Shifted-CIR no-arbitrage term structure model, where the instantaneous interest rate is given by the sum of a constant lower bound and a non-negative CIR-like process, allows for negative yields and benefits from similar tractability of the original CIR model. Based on the U.S. and German yield curve data, the Vasicek and Shifted-CIR specifications, both considering constant and time-varying risk premia, are compared in terms of information criteria and forecasting ability. Information criteria prefer the Shifted-CIR specification to models based on the Vasicek process. It also provides similar or better in-sample and out-of-sample forecasting ability of future yield curve movements. Introducing a time variation of the interest rate risk premium in no-arbitrage one-factor term structure models is instead not recommended, as it provides worse information criteria and forecasting performance.

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INTRODUCTION AND LITERATURE REVIEW

The seminal contributions by Vasicek (1977) and Cox, Ingersoll, & Ross (1985) spawned a vast literature on no-arbitrage dynamic term structure models. The original affine one-factor models have inspired specifications accounting for multiple factors, which are either exclusively yield-curve related, (e.g., Duffie & Kan, 1996; Dai & Singleton, 2000; Dai & Singleton, 2002; Duffee, 2002: Joslin, Singleton, & Zhu, 2011) or also macro-based (e.g., Joslin, Le, & Singleton, 2013; Joslin, Priebsch, & Singleton, 2014).

Flexible multi-factor models, especially when allowing for time-varying risk premia, are not exempt by overfitting issues. Duffee (2011), Feldhütter, Larsen, Munk, & Trolle (2012), and Sarno, Schneider, & Wagner (2016) highlight significant difficulties engendered by estimation risk, which leads to poor performance in forecasting yield curve movements and out-of-sample portfolio construction. In particular, they suggest caution in the use of specifications accounting for time-varying risk premia and recommend parsimony in the choice of the number of factors.

One-factor models are still of interest, as they are often preferred for their parsimony and analytical tractability in works that do not

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exclusively focus on the pricing of the yield curve. For instance, they are used in structural models of corporate claim valuation (e.g., Longstaff & Schwartz, 1995; Briys & De Varenne, 1997; Collin-Dufresne & Goldstein, 2001; Ju & Ou-Yang, 2006; Martellini, Milhau, & Tarelli, 2018). As another example, the Cox-Ingersoll-Ross (CIR) model has been extensively used to price American options under stochastic interest rates (e.g., Medvedev & Scaillet, 2010; Boyarchenko & Levendorskiĭ, 2013).

A drawback of the original CIR model is that it is not compatible with negative interest rates, which are nowadays a common fact, especially in Europe. To overcome this issue and preserve the analytical tractability, the Vasicek interest rate process, which is compatible with negative rates, is often preferred. To mention a recent contribution to the American option pricing literature, Battauz & Rotondi (2019) justify the choice of a Vasicek interest rate process precisely for this reason. However, the Gaussian conditional distribution of interest rates under a Vasicek-type instantaneous interest rate process entails unlikely forecast distributions of yields when interest rates are around or slightly below zero. For example, the estimation of the Vasicek model performed in the present paper, conditional on the last observation of the German government bond yields available (July 1, 2019), leads to an estimate of the end-of-sample instantaneous interest rate of –0.44%, in line with the ECB deposit rate of –0.4% on that date, but the conditional probabilities of the instantaneous interest rate being, respectively, below –1% or –2% in one-year time are unreasonably high (28% and 9%).

The aforementioned issue is common to all multi-factor affine term structure models where the state variables follow a multivariate Ornstein-Uhlenbeck process and has led several authors to study the models where the observed short-term rate is given by the maximum between a shadow rate, which distribution is conditionally Gaussian, and a lower-bound rate (e.g., Krippner, 2013; Wu & Xia, 2016; Chung, Hui, & Li, 2017). However, because of the non-linearity introduced, these models are not as easily tractable as affine models from an analytical and econometric perspectives.

This paper studies the empirical performance of a Shifted-CIR (S-CIR) model, where the instantaneous interest rate is given by the sum of a lower-bound level and a strictly positive state variable that follows a CIR-type process, entailing a non-zero (possibly negative) lower-bound to the interest rates. This specification is a particular case of more flexible models, such as the multivariate essentially affine models in Duffee (2002) or the semi affine square-root models in Duarte (2004), as well as the CIR++ specification in Brigo & Mercurio (2007). Importantly, differing from these models, the S-CIR specification is very parsimonious and benefits from the same analytical tractability of the original CIR model.

Different specifications of the S-CIR model, allowing for either constant or time-varying risk premia, are compared to the corresponding Vasicek-type specifications. The models are estimated using the U.S. and German data, and the comparison is performed according to information criteria, as well as in terms of in- and out-of-sample forecasting ability. As opposed to a Vasicek specification, the S-CIR model with a constant risk premium is found to be preferred by information criteria, to have similar or better in- and out-of-sample yield forecasting ability and, finally, to partially explain the empirically-observed time-varying volatility of interest rates.

This paper also contributes to the literature studying the potential out-of-sample benefits of predicting the time-variation of bond risk premia through no-arbitrage term structure models (e.g., Sarno, Schneider, & Wagner, 2016), which is typically focused on multivariate models. In the context of univariate specifications, the present study shows that a time-varying specification for the interest rate risk premium is not recommended in terms of model quality and forecasting ability for both the Vasicek and the S-CIR models, finding the introduction of time-varying risk premia to be particularly detrimental in the case of Vasicek-based specifications.

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The remainder of the paper is organized as follows. Section 1 introduces the notations and pricing formulae of the models considered, provides the details of the dataset used, and describes the estimation methodology. Section 2 presents the in- and out-of-sample empirical findings, which are summarized and discussed in Section 3. The final section concludes.

1. METHOD

This section provides the theoretical setting of the models considered, then describes the data and the methodology adopted in the empirical analysis.

1.1. Model specification and bond pricing

In the following, the notations used for the Vasicek and S-CIR model specifications are introduced, and the corresponding closed-form bond pricing formulae are provided.

1.1.1. Vasicek

Consider an Ornstein-Uhlenbeck process for the P-dynamics of the state variable x_t:

$$dx_t = \theta\left(\overline{x} - x_t\right)dt + \sigma_x dz_t, \tag{1}$$

where z_t is a standard one-dimensional Wiener process defined for $t \geq 0$. Express the instantaneous interest rate as an affine function of the underlying state variable, i.e., $R_t = r_0 + r_1 x_t$, where r_0 and r_1 are constants. If the underlying process in (1) is unobservable, any possible parametrization of r_0 and $r_1 \neq 0$ leads to observationally equivalent specifications of the model[1]. The most common parametrization, corresponding to the model by Vasicek (1977), is obtained by taking $r_0 = 0$ and $r_1 = 1$. This means that the interest rate is coincident with the underlying state variable, i.e., $R_t = x_t$.

Given a state-dependent market price of risk

$$\Lambda_t = \lambda_0 + \lambda_1 x_t, \tag{2}$$

as a consequence of the Girsanov's theorem, the Q-dynamics for x_t is $dx_t = \theta^Q\left(\overline{x}^Q - x_t\right)dt + \sigma_x dz_t^Q$,

where $\overline{x}^Q = \dfrac{\theta\overline{x} - \sigma_x\lambda_0}{\theta + \sigma_x\lambda_1}$ and $\theta^Q = \theta + \sigma_x\lambda_1$.

Denoting with $B(t,\tau)$ the time t price of a zero-coupon bond maturing at $T = t + \tau$ and imposing no-arbitrage restrictions, the following Feynman-Kac equation is obtained:

$$\frac{\partial B}{\partial t} + \frac{\partial B}{\partial x}\theta^Q\left(\overline{x}^Q - x_t\right) + \frac{1}{2}\frac{\partial^2 B}{\partial x^2}\sigma_x^2 = Bx_t,$$

where the boundary condition at maturity implies $B(t,0) = 1$. The analytic solution to the pricing problem is:

$$B(t,\tau) = e^{-a(\tau) - b(\tau)x_t},$$

where

$$a(\tau) = \left(\tau - \frac{1 - e^{-\theta^Q\tau}}{\theta^Q}\right)\overline{x}^Q - $$
$$- \frac{\sigma_x^2}{2\left(\theta^Q\right)^2}\left(\tau - 2\frac{1 - e^{-\theta^Q\tau}}{\theta^Q} + \frac{1 - e^{-2\theta^Q\tau}}{2\theta^Q}\right),$$

$$b(\tau) = \frac{1 - e^{-\theta^Q\tau}}{\theta^Q}.$$

The zero-coupon yield is thus equal to:

$$y(t,\tau) = -\frac{1}{\tau}\log B(t,\tau) = \tilde{a}(\tau) + \tilde{b}(\tau)x_t, \tag{3}$$

where $\tilde{a}(\tau) = \dfrac{a(\tau)}{\tau}$ and $\tilde{b}(\tau) = \dfrac{b(\tau)}{\tau}$.

The expected excess return of a zero-coupon bond, equal to the product of the diffusion coefficient of the bond return process with the market price of risk Λ_t in (2), is affine in the state variable x_t:

$$\frac{1}{dt}\mathrm{E}_t\left[\frac{dB}{B}\right] - R_t = \frac{1}{B}\frac{\partial B}{\partial x}\sigma_x\Lambda_t = -b(\tau)\sigma_x\left(\lambda_0 + \lambda_1 x_t\right). \tag{4}$$

1 See Dai and Singleton (2000) for a formal analysis of the invariant transformations applicable to affine term structure models that lead to observationally equivalent specifications.

1.1.2. Shifted-CIR

Consider a CIR-type P-dynamics for the underlying state variable x_t:

$$dx_t = \theta\left(\overline{x} - x_t\right)dt + \sigma_x \sqrt{x_t}\, dz_t.$$

In the original CIR model, the instantaneous interest rate is $R_t = x_t$, consider instead an S-CIR specification where:

$$R_t = \underline{r} + x_t.$$

This adds a degree of freedom in the specification of the model[2], as the conditional volatility of the interest rate process is not zero when $R_t = 0$, but it is zero at the level $\underline{r}$, which represents the lower bound of the interest rate process[3].

Introduce a state-dependent market price of risk as given by

$$\Lambda_t = \frac{\lambda_0}{\sqrt{x_t}} + \lambda_1 \sqrt{x_t}. \qquad (5)$$

As a consequence of the Girsanov's theorem, the Q-dynamics for x_t is $dx_t = \theta^Q\left(\overline{x}^Q - x_t\right)dt + \sigma_x \sqrt{x_t}\, dz_t^Q$, where $\overline{x}^Q = \dfrac{\theta\overline{x} - \sigma_x \lambda_0}{\theta + \sigma_x \lambda_1}$ and $\theta^Q = \theta + \sigma_x \lambda_1$.

Note that the drift term is affine in the state variables both under the P- and the Q-measure[4].

Denoting with $B(t,\tau)$ the time t price of a zero-coupon bond maturing at $T = t + \tau$ and imposing no-arbitrage restrictions, the following Feynman-Kac equation is obtained:

$$\frac{\partial B}{\partial t} + \frac{\partial B}{\partial x}\theta^Q\left(\overline{x}^Q - x_t\right) + \frac{1}{2}\frac{\partial^2 B}{\partial x^2}\sigma_x^2 x_t = B\left(\underline{r} + x_t\right),$$

where the boundary condition at maturity implies $B(t,0) = 1$. The analytic solution to the pricing problem is:

$$B(t,\tau) = e^{-a(\tau) - b(\tau)x_t},$$

where

$$a(\tau) = \underline{r}\tau - \frac{2\theta^Q \overline{x}^Q}{\sigma_x^2}\log\left(\frac{2he^{\frac{\theta^Q h}{2}\tau}}{2h + \left(\theta^Q + h\right)\left(e^{h\tau} - 1\right)}\right),$$

$$b(\tau) = \frac{2\left(e^{h\tau} - 1\right)}{2h + \left(\theta^Q + h\right)\left(e^{h\tau} - 1\right)} \quad \text{and}$$

$h = \sqrt{\left(\theta^Q\right)^2 + 2\sigma_x^2}$. Considering $a(\tau)$ and $b(\tau)$ just provided, the zero-coupon yield is given by the same expression as in (3). Similarly, the expected excess return of a zero-coupon bond is given by (4), and thus, as for the Vasicek model, it is affine in the state variable x_t. This result justifies the specification for the market price of risk Λ_t in (5).

1.2. Data

The models are estimated using the U.S. monthly data from August 1971 until June 2019 and German data from January 1960 until July 2019. For the U.S. dataset, zero-coupon nominal yields for the following maturities are considered: 3 and 6 months, and from 1 to 10 years by 1-year steps. The 3- and 6-month yields were obtained from the Treasury Bill rates, available on the Federal Reserve Economic Data (FRED) website[5] (series GS3M and GS6M). The other nominal zero-coupon yields are the series fitted by Gürkaynak, Sack, & Wright (2007)[6], available on the website of the Federal Reserve Board[7]. The model is also estimated using German nominal bond yields for the maturities of 3 months and 10 years, which are also available on the FRED website (series IR3TIB01DEM156N and IRLTLT01DEM156N).

2 As the process x_t is unobservable, similarly to the Vasicek model previously proposed, any other affine specification of the instantaneous interest rate, i.e., $R_t = r_0 + r_1 x_t$, would lead to an observationally equivalent model as long as $r_1 > 0$.

3 The dynamics of the interest rate process could be equivalently written as $dR_t = \theta\left(\overline{r} - R_t\right)dt + \sigma_r \sqrt{R_t - \underline{r}}\, dz_t$.

4 In the context of multi-factor term structure models, Duarte (2004) introduces a specification of the market prices of risk such that the P-drift of the state variables is a semi-affine (non-linear) function of the state variables themselves. The additional flexibility comes at the cost of a harder analytical and econometric tractability.

5 https://research.stlouisfed.org/fred2/

6 The yield curves are fitted for maturities between 1 and 30 years. As the authors discourage extrapolations outside of this range, for the short end of the yield curve their dataset is complemented using Treasury Bill rates.

7 https://www.federalreserve.gov/pubs/feds/2006/200628/200628abs.html

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1.3. Estimation methodology

The models are estimated by maximum likelihood from the time series of historical bond yields corresponding to several different maturities. The time t observation of the bond yield for the maturity τ_i is denoted with $y^o(t,\tau_i)$. Along the lines of Duffee (2002), it is assumed that a number of bond yields, equal to the number of state variables d in the model, are perfectly observed. The other $n-d$ zero-coupon yields are instead allowed to be observed with errors. As there is only one state variable in the specifications considered, the yield corresponding to the shortest maturity $(\tau_1 = 0.25)$ is chosen to be perfectly observed, which entails that the state variable x_t can be inferred from the observation of $y^o(t,\tau_1)$:

$$x_t = \frac{y^o(t,\tau_1) - \tilde{a}(\tau_1)}{\tilde{b}(\tau_1)}.$$

The other $n-1$ yields are observed with Gaussian observation errors $\eta_{i,t}$, which are assumed to be i.i.d. both in time series and cross-sectionally, with variance $\sigma^2_{\tau_i}$:

$$y^o(t,\tau_i) = y(t,\tau_i) + \eta_{i,t} = \tilde{a}(\tau_i) + \tilde{b}(\tau_i)x_t + \eta_{i,t},$$

$$\eta_{i,t} \sim N\left(0,\sigma^2_{\tau_i}\right),\ (i = 2,\ldots,n).$$

The p.d.f. of the errors is $f(\eta_{i,t}) = \frac{1}{\sqrt{2\pi\sigma^2_{\tau_i}}} e^{-\frac{\eta^2_{i,t}}{2\sigma^2_{\tau_i}}}$.

The advantage with respect to a semi-affine specification (Duarte, 2004) is that the likelihood function can be analytically characterized, as the conditional distribution $f(x_t|x_{t-1})$ can be expressed in closed form.

For the Vasicek specification, it is:

$$f(x_t|x_{t-1}) = \frac{1}{\sqrt{2\pi\sigma^2_{\Delta t}}} e^{-\frac{\left(x_t - (\alpha + \beta x_{t-1})\right)^2}{2\sigma^2_{\Delta t}}},$$

where

$$\alpha = \overline{x}\left(1 - e^{-\theta\Delta t}\right),\ \beta = e^{-\theta\Delta t},\ \sigma^2_{\Delta t} = \frac{1 - e^{-2\theta\Delta t}}{2\theta}\sigma^2_x.$$

For the S-CIR specification, it is:

$$f(x_t|x_{t-1}) = ce^{-u_t - v_t}\left(\frac{v_t}{u_t}\right)^{\frac{q}{2}} I_q\left(2\sqrt{u_t v_t}\right),$$

where

$$c = \frac{2\theta}{\sigma^2_x\left(1 - e^{-\theta\Delta t}\right)},\ q = \frac{2\theta\overline{x}}{\sigma^2_x} - 1,\ u_t = cx_{t-1}e^{-\theta\Delta t},$$

$v_t = cx_t$. $I_q(.)$ is the modified Bessel function of the first kind of order q. As in Duffee (2002), the conditional time t log-likelihood is:

$$\ell_t = -\log\left|\tilde{b}(\tau_1)\right| + \log f(x_t|x_{t-1}) + \sum_{i=2}^{n} \log f(\eta_{i,t}).$$

The global log-likelihood $L = \sum_{t=1}^{T} \ell_t$ can be then numerically maximized with respect to the parameter space $\Theta = \left\{\overline{x}, \theta, \sigma_x, \lambda_0, \lambda_1, \underline{r}, \sigma_{\tau_2}, \ldots, \sigma_{\tau_n}\right\}$.

A Nelder-Mead simplex algorithm is used and, to help the algorithm converge for the S-CIR specifications, the parameters are first estimated assuming that x_t coincides with the shortest-maturity yield shifted by $-\underline{r}$. These first-step estimates are then used as starting point for the final estimation of the parameters.

2. RESULTS AND DISCUSSION

This section first discusses the findings obtained through an in-sample analysis, where the models are estimated using the full sample available, and then reports the results from an out-of-sample analysis.

2.1. In-sample estimation

Table 1 shows the in-sample parameter estimates for the four different specifications considered, Vasicek and S-CIR, either considering a constant risk premium (CRP), where the restriction $\lambda_1 = 0$ is imposed, or a time-varying risk premium (TVRP), where $\lambda_1 \neq 0$. These are obtained using the full sample available for each of two countries. Most estimates are statistically significant, with the exception in some cases of the long-run mean $\overline{x}$. Allowing for a TVRP reduces the significance of the estimates of the speed of mean reversion θ. Under a CRP, the constant market price of risk λ_0 is always statistically significant. In contrast, its significance is particularly reduced for the Vasicek TVRP specification, but not for the S-CIR TVRP. λ_1 relates the time variation of the market price of risk to the state variable. Its standard error is large under all TVRP models, which is common in the estimation of affine term structure models. As expected, the lower bound $\underline{r}$ for the S-CIR specifications is negative, as the 3-month yield takes slightly negative values in at least one date in both country samples. Furthermore, the estimates for $\underline{r}$ are statistically significant and identical between the CRP and TVRP specifications for the same country. Being the shortest-maturity yield perfectly observed by assumption, the standard deviations of the observation errors are increasing in maturity. Furthermore, they are very similar across model specifications.

Table 1. In-sample parameter estimates

(a) U.S. data (1971–2019)

Parameter	Vasicek CRP	Vasicek TVRP	S-CIR CRP	S-CIR TVRP
$\bar{x}$	0.0375	0.0427	0.0520	0.0401
	(0.0454)	(0.0202)	(0.0130)	(0.0097)
θ	0.0601	0.1368	0.0506	0.0682
	(0.0019)	(0.0779)	(0.0016)	(0.0174)
σ_x	0.0189	0.0189	0.0709	0.0710
	(0.0006)	(0.0007)	(0.0036)	(0.0043)
λ_0	−0.3252	−0.1352	−0.0724	−0.0700
	(0.1449)	(0.2406)	(0.0087)	(0.0121)
λ_1	–	−4.0517	–	−0.2614
	–	(4.0980)	–	(0.2478)
$\underline{r}$	–	–	−0.0012	−0.0012
	–	–	(0.0005)	(0.0006)
$\sigma_{0.5}$	0.0025	0.0025	0.0025	0.0025
σ_1	0.0067	0.0067	0.0068	0.0067
σ_2	0.0082	0.0082	0.0083	0.0083
σ_3	0.0093	0.0093	0.0093	0.0093
σ_4	0.0101	0.0101	0.0101	0.0101
σ_5	0.0106	0.0106	0.0106	0.0106
σ_6	0.0110	0.0110	0.0110	0.0111
σ_7	0.0114	0.0114	0.0114	0.0114
σ_8	0.0117	0.0117	0.0118	0.0117
σ_9	0.0120	0.0120	0.0120	0.0120
σ_{10}	0.0122	0.0122	0.0123	0.0123
L	23031.53	23032.02	23318.35	23318.58
N_{par}	15	16	16	17
AIC	−46033.06	−46032.04	−46604.69	−46603.15
BIC	−45967.78	−45962.40	−46535.05	−46529.16

(b) German data (1960–2019)

Parameter	Vasicek CRP	Vasicek TVRP	S-CIR CRP	S-CIR TVRP
$\bar{x}$	0.0324	0.0345	0.0199	0.0282
	(0.0286)	(0.0255)	(0.0108)	(0.0244)
θ	0.0613	0.0729	0.0528	0.0286
	(0.0045)	(0.0566)	(0.0046)	(0.0255)
σ_x	0.0135	0.0135	0.0529	0.0530
	(0.0004)	(0.0004)	(0.0014)	(0.0014)
λ_0	−0.3083	−0.2688	−0.0910	−0.0962
	(0.1301)	(0.2318)	(0.0114)	(0.0283)
λ_1	–	−0.8669	–	0.4677
	–	(4.2034)	–	(0.4751)
$\underline{r}$	–	–	−0.0045	−0.0045
	–	–	(0.0001)	(0.0001)
σ_{10}	0.0123	0.0123	0.0123	0.0123
L	5076.28	5076.30	5315.44	5315.57
N_{par}	5	6	6	7
AIC	−10142.56	−10140.61	−10618.89	−10617.14
BIC	−10119.71	−10113.18	−10591.46	−10585.15

Note: The tables show the maximum-likelihood estimates of the model parameters. The standard errors of the estimates are in brackets. The likelihood value L, the AIC, and the BIC criteria are also reported. The sample period for the U.S. market (panel (a)) runs from August 1971 until June 2019. The sample period for the German market (panel (b)) runs from January 1960 until July 2019.

2.2. In-sample model selection

Table 1 also reports the value of the likelihood function L, as well as the values for the Akaike information criterion (AIC) and the Bayesian information criterion (BIC). Information theory recommends the choice of the model with the lowest value for these two criteria. For both AIC and BIC, and for both datasets, the preferred model is the S-CIR CRP. Indeed, the S-CIR specification entails a significant increase of the likelihood function with respect to the Vasicek model, which more than counterbalances the increase in the number of parameters. Conversely, allowing for TVRP, for both the Vasicek and the S-CIR specifications, entails an increase of model complexity that is not counterbalanced by the tiny increase of L.

2.3. In-sample moment matching

To provide further evidence on the goodness of fit of the models proposed, Table 2 reports the summary statistics on the average values and volatilities of the yields in the dataset, as well as their model-im-

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plied counterparts. For the U.S. dataset, all models match rather well the average bond yields in the data, with an almost perfect matching at the short end of the yield curve and for the maturities from 5 to 7 years. For the German dataset, where the maturities observed are only two, the matching of the means is nearly perfect. The picture is slightly different for what concerns the volatilities. For the U.S. dataset, the S-CIR specifications underestimate the volatilities at the very short end, but the matching is rather good (within about 0.1%) for maturities of 1 year or longer. The Vasicek specifications instead overestimate the volatilities across the board. The picture is similar for the German dataset, where the S-CIR specifications underestimate the volatility for the 3-month maturity and overestimate it for the 10-year maturity. At the same time, the Vasicek model fits the short-end volatility very well, but strongly overestimates it at the long end. Across the board, in this analysis, small differences are entailed by accounting for time-varying risk premia in comparison to the specifications based on constant risk premia.

Table 2. Data and model-implied averages and volatilities of bond yields

(a) U.S. data (1971–2019)

Yield	Data	Vasicek CRP	Vasicek TVRP	S-CIR CRP	S-CIR TVRP
Avg. 6M bond yield	4.74%	4.74%	4.74%	4.74%	4.74%
Avg. 6M bond yield	4.92%	4.80%	4.80%	4.80%	4.80%
Avg. 1Y bond yield	5.13%	4.93%	4.93%	4.93%	4.93%
Avg. 2Y bond yield	5.37%	5.18%	5.18%	5.17%	5.17%
Avg. 3Y bond yield	5.56%	5.41%	5.41%	5.40%	5.40%
Avg. 4Y bond yield	5.73%	5.62%	5.62%	5.61%	5.61%
Avg. 5Y bond yield	5.88%	5.81%	5.81%	5.81%	5.80%
Avg. 6Y bond yield	6.01%	5.99%	5.99%	5.99%	5.99%
Avg. 7Y bond yield	6.12%	6.16%	6.16%	6.16%	6.16%
Avg. 8Y bond yield	6.23%	6.31%	6.31%	6.32%	6.32%
Avg. 9Y bond yield	6.31%	6.46%	6.46%	6.47%	6.47%
Avg. 10Y bond yield	6.39%	6.59%	6.59%	6.61%	6.61%
Vol. 3M bond yield	1.74%	1.87%	1.88%	1.54%	1.54%
Vol. 6M bond yield	1.71%	1.86%	1.87%	1.53%	1.53%
Vol. 1Y bond yield	1.62%	1.83%	1.84%	1.51%	1.51%
Vol. 2Y bond yield	1.48%	1.78%	1.78%	1.47%	1.47%
Vol. 3Y bond yield	1.37%	1.73%	1.73%	1.43%	1.43%
Vol. 4Y bond yield	1.29%	1.68%	1.68%	1.39%	1.39%
Vol. 5Y bond yield	1.23%	1.63%	1.64%	1.35%	1.35%
Vol. 6Y bond yield	1.19%	1.59%	1.59%	1.30%	1.31%
Vol. 7Y bond yield	1.16%	1.54%	1.55%	1.26%	1.27%
Vol. 8Y bond yield	1.14%	1.50%	1.50%	1.22%	1.23%
Vol. 9Y bond yield	1.12%	1.46%	1.46%	1.18%	1.19%
Vol. 10Y bond yield	1.11%	1.42%	1.42%	1.14%	1.15%

(b) German data (1960–2019)

Yield	Data	Vasicek CRP	Vasicek TVRP	S-CIR CRP	S-CIR TVRP
Avg. 10Y bond yield	4.59%	4.59%	4.59%	4.59%	4.59%
Avg. 10Y bond yield	5.74%	5.73%	5.73%	5.73%	5.74%
Vol. 3M bond yield	1.34%	1.34%	1.34%	1.18%	1.18%
Vol. 10Y bond yield	0.66%	1.01%	1.01%	0.89%	0.89%

Note: The first column of the tables reports the average values of the yields for the different maturities available in the datasets, as well as their annualized volatilities. The other columns, for the four model specifications considered, report the corresponding model-implied quantities. The volatilities for the S-CIR specifications are evaluated at the average value of the model-implied state variable x_t. The sample period for the U.S. market (panel (a)) runs from August 1971 until June 2019. The sample period for the German market (panel (b)) runs from January 1960 until July 2019.

2.4. In-sample forecasting ability

Dynamic term structure models can be used to forecast yield variations. For all the models considered, the time t expected yield variation over an interval Δt is:

$$E_t\left[y(t+\Delta t,\tau)-y(t,\tau)\right]= \\ =\tilde{b}(\tau)(\bar{x}-x_t)\left(1-e^{-\theta\Delta t}\right).$$

To assess the in-sample forecasting ability of a model, these variations can be compared with the corresponding observed yield variations, $y^o(t+\Delta t,\tau)-y^o(t,\tau)$. Panels (a) and (b) of Figure 1 show the forecasting root-mean-squared errors (RMSE) for a forecasting period $\Delta t = 6$ months relative to the RMSE under the hypothesis that the yields follow a random walk (RW), i.e., when the yield variation forecast is null. When the ratio is lower than 1, the model improves the forecasts obtained under the RW hypothesis. The fact that the ratio is often greater than 1 is not surprising, as previous evidence (e.g., Duffee, 2002) showed that even 3-factor affine term structure models have weak forecasting ability[8]. However, for both samples, all models seem to have better forecasting ability than the RW for short-maturity yields. While all models (except the S-CIR TVRP for the German dataset) perform slightly worse than the RW for long-maturity yields, especially for

8 Despite the often weak forecasting ability of yield variations for specific maturities, note that no-arbitrage term structure models, as opposed to the RW hypothesis, have the important advantage of being able to describe the joint movements of yields corresponding to different maturities.

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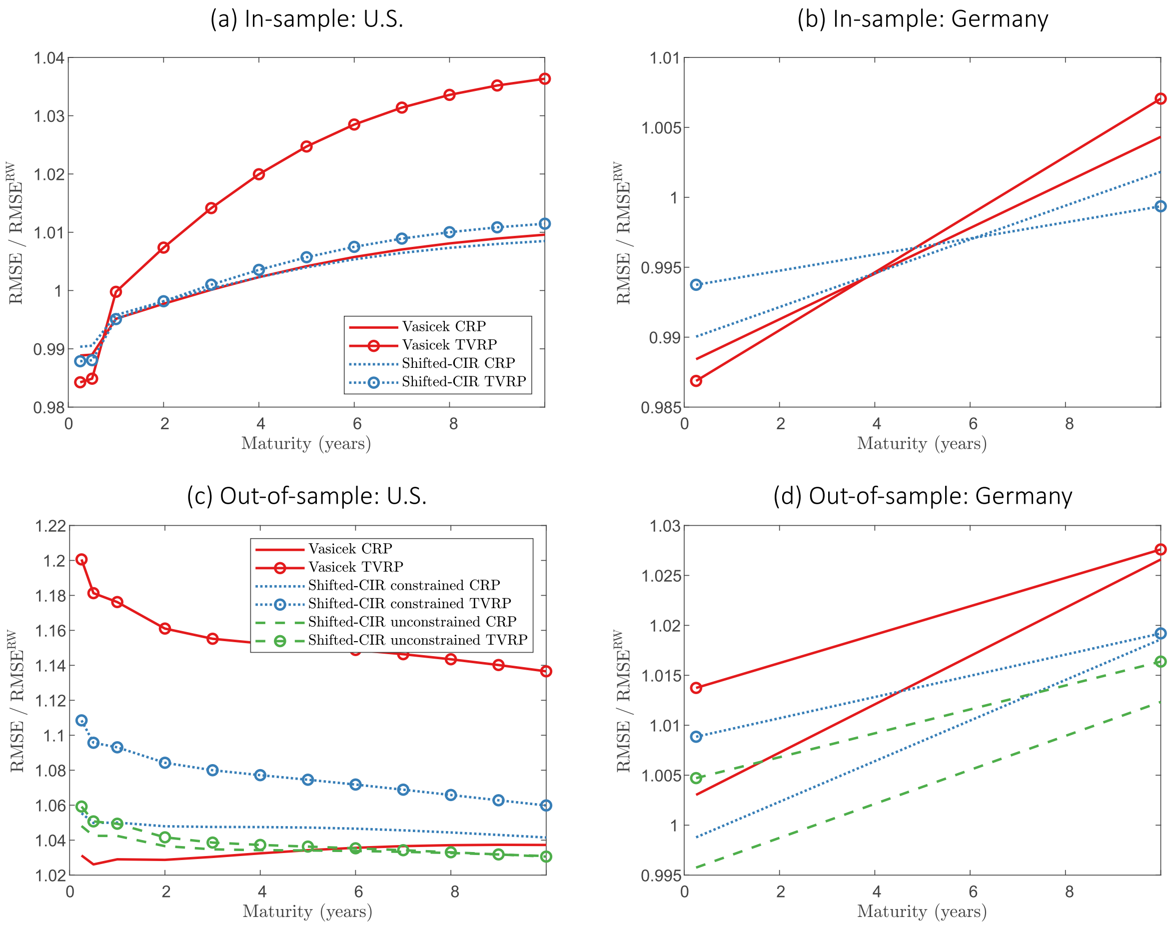

Note: For the different maturities considered (3 months to 10 years), the figure shows the ratio between the root-mean-squared errors of yield variations forecast obtained with the different models considered (RMSE) and the root-mean-squared errors of yield variations obtained under the random walk hypothesis $\left(\text{RMSE}^{\text{RW}}\right)$. The forecasting horizon is $\Delta t = 6$ months.

Figure 1. In-sample and out-of-sample 6-month-ahead yield variation forecasting ability in comparison to the random walk hypothesis

the U.S. dataset, the Vasicek TVRP model performs significantly worse than all the others. For what concerns the German dataset, in line with the results obtained on moment matching, the Vasicek specification allows better forecasting yield variations on the short end on the yield curve, while the S-CIR specification performs relatively better for long-maturity yields.

2.5. In-sample volatility forecasting

An interesting feature of CIR-based interest rate models is that the conditional volatility of the instantaneous interest rate is time-varying (Cox, Ingersoll, & Ross, 1985). In particular, in the S-CIR specification proposed, the conditional variance over a time interval Δt is:

$$Var_t\left[R_{t+\Delta t} - R_t\right] = Var_t\left[R_{t+\Delta t}\right] =$$

$$= \bar{x}\frac{\sigma_x^2}{2\theta}\left(1-e^{-\theta\Delta t}\right)^2 + \frac{\sigma_x^2}{\theta}\left(e^{-\theta\Delta t} - e^{-2\theta\Delta t}\right)x_t.$$

Conversely, the conditional variance under the Vasicek-based models is constant:

$$Var_t\left[R_{t+\Delta t} - R_t\right] = Var_t\left[R_{t+\Delta t}\right] = \frac{\sigma_x^2}{2\theta}\left(1-e^{-2\theta\Delta t}\right).$$

Panels (a) and (b) of Figure 2 show the model-implied in-sample annualized conditional volatilities of the instantaneous interest rate and the empirical annualized volatility of the 3-month rate observed over 36 months following the forecasting date. Note that, as CRP and TVRP models of the

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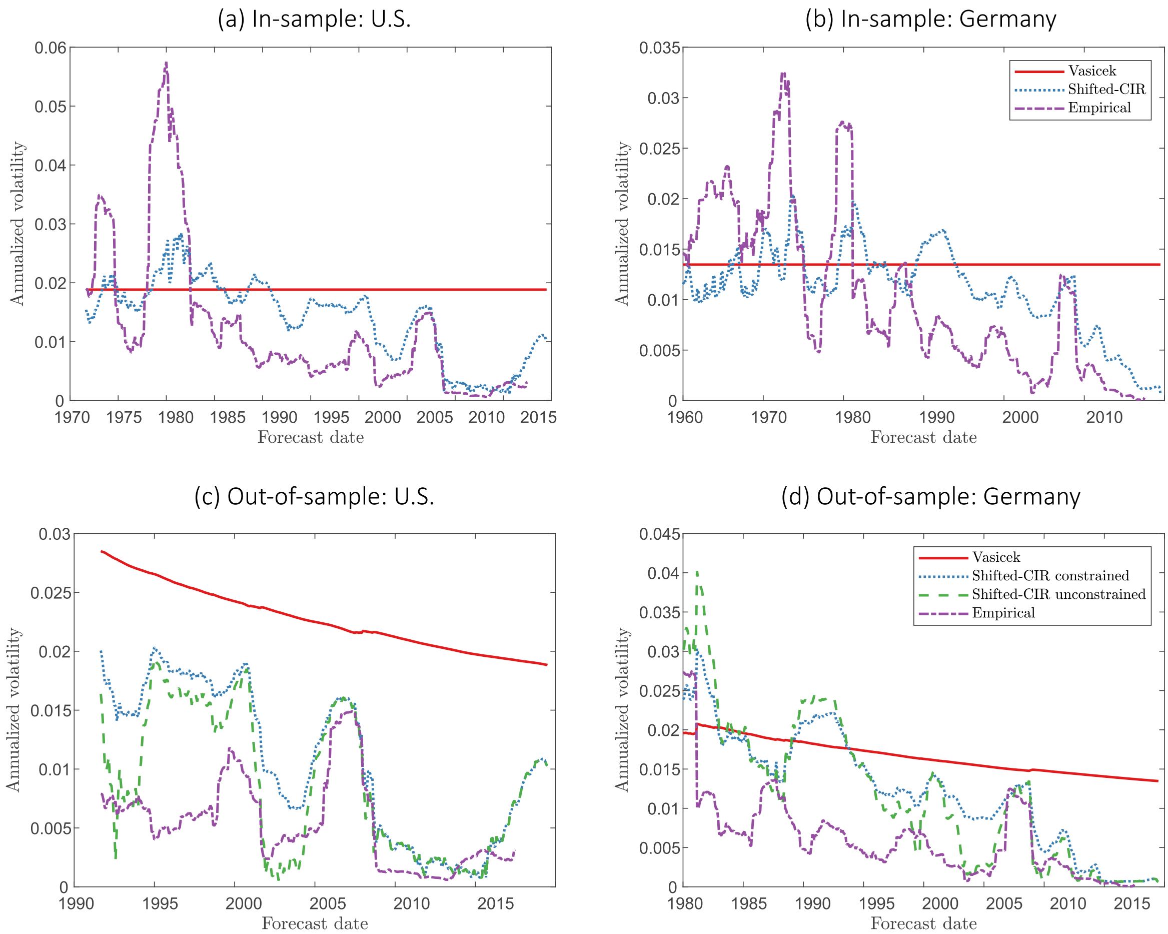


Note: The figure shows the monthly annualized volatility forecasts obtained with the Vasicek and Shifted-CIR models. As constant risk premium and time-varying risk premium specifications of the same model type provide indistinguishable forecasts, only one line per type is reported. The empirical annualized volatility of the 3-month rate over 36 months following the forecasting date is also shown.

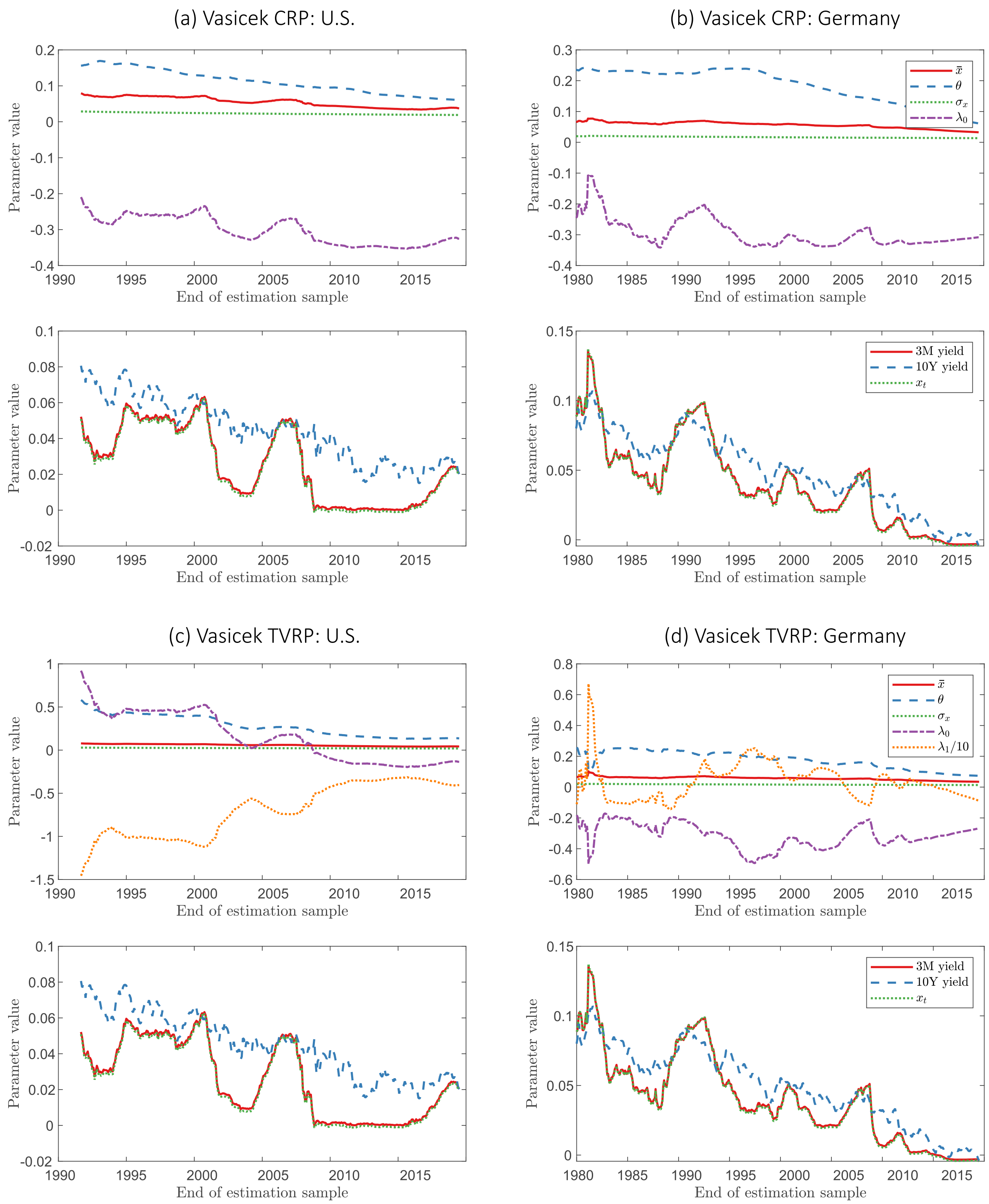
Figure 2. In-sample and out-of-sample short-term rate volatility forecasting

same type provide indistinguishable forecasts, only one line per type is plotted. As can be noticed, the S-CIR model shows some forecasting ability of the short-term rate volatility. The R^2 obtained from regressing the realized volatility on the S-CIR volatility forecasts are 45.5% (U.S.) and 30.2% (Germany), while the R^2 is obviously 0 when regressing the realized volatility on the Vasicek forecasts.

2.6. Out-of-sample estimation

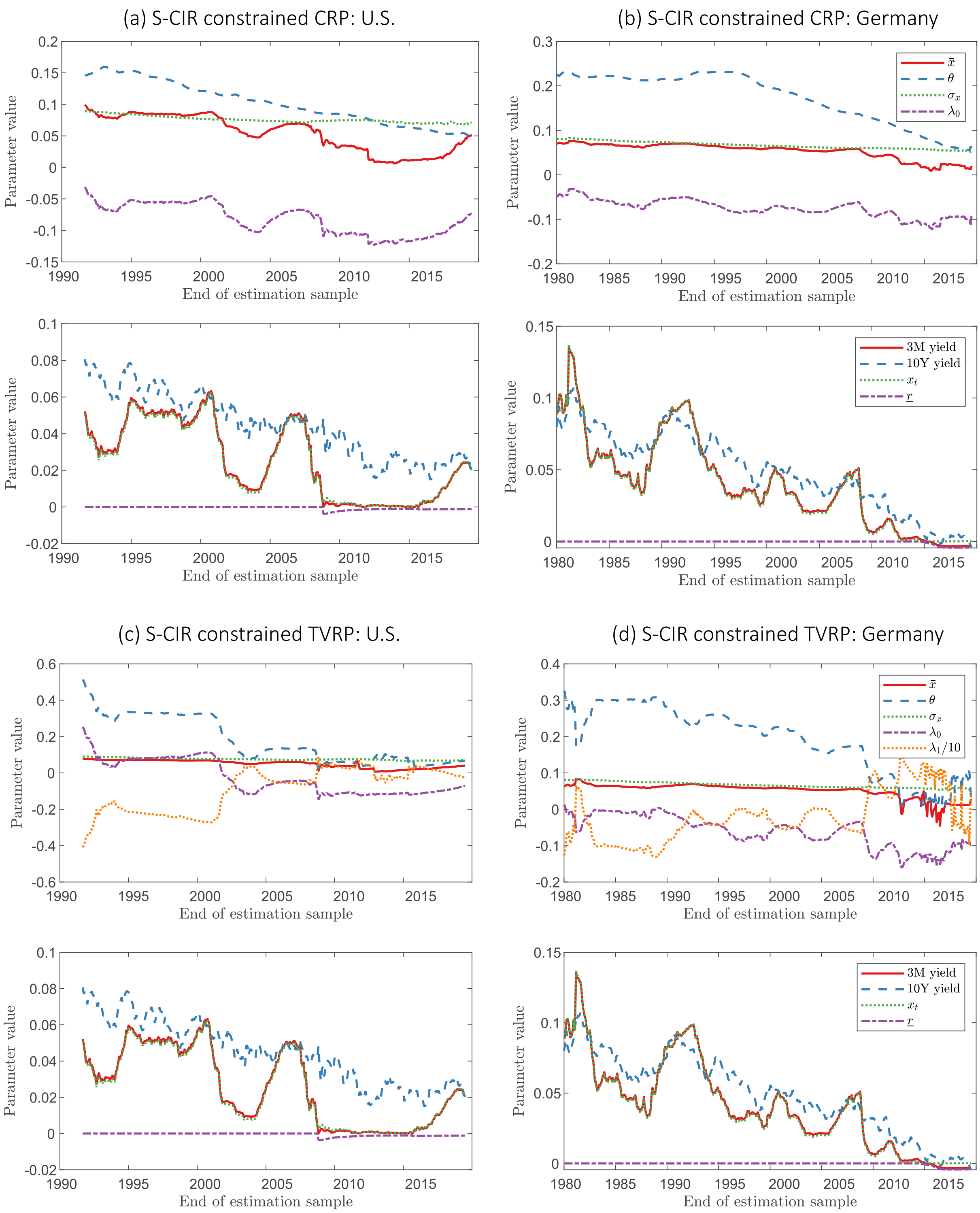
The out-of-sample analysis is performed by estimating the models over an expanding window starting from the beginning of the sample and having an initial sample size of 240 monthly observations. Monthly increments then expand the window up to the end of the sample. In the out-of-sample analysis, the S-CIR specifications are estimated under two different cases: a constrained one, where the lower bound is non-positive $(\underline{r} \leq 0)$, and an unconstrained one, where $\underline{r}$ is allowed to be positive. The first one has the main purpose of allowing a CIR-like model to accommodate for negative interest rates, while the second one is more flexible, as the lower bound $\underline{r}$ can be positive if this best fits the data[9]. Figure 3 shows the time series of the estimates obtained for the Vasicek specifications, while Figures 4 and 5, respectively, refer to the S-CIR constrained and

9 For the in-sample analysis, considering two different cases is not necessary, as the maximum-likelihood estimates obtained for $\underline{r}$ are negative (Table 1), and the constraint would not affect.



Note: The figure shows out-of-sample parameter estimates obtained for the Vasicek specifications with constant (CRP) and time-varying risk premia (TVRP). The initial estimation window is made of 240 monthly observations, starting on August 1971 for the U.S. dataset and on January 1960 for the German dataset. The X-axis values represent the end of the estimation window. x_t represents the end-of-estimation-period value of the unobservable state variable.

Figure 3. Out-of-sample parameter estimates for the Vasicek models



Note: The figure shows out-of-sample parameter estimates obtained for the S-CIR specifications with constant (CRP) and time-varying risk premia (TVRP). The lower bound to the interest rate $\underline{r}$ is constrained to be non-positive. The initial estimation window is made of 240 monthly observations, starting on August 1971 for the U.S. dataset and on January 1960 for the German dataset. The X-axis values represent the end of the estimation window. x_t represents the end-of-estimation-period value of the unobservable state variable.

Figure 4. Out-of-sample parameter estimates for the S-CIR models with non-positive lower bound

unconstrained specifications. Panels (a) and (b) refer to the cases with constant risk premia, respectively for U.S. and Germany, while panels (c) and (d) represent the cases with time-varying risk premia. Unsurprisingly, the time series of parameter estimates are more stable under CRP than under TVRP, especially for what concerns the estimates for λ_0 and λ_1. In some cases, under TVRP, also the estimates for the long-run mean $\bar{x}$ and for the speed of mean reversion θ are more unstable than under CRP. However, as can be noticed in Figures 4 and 5, the estimates for $\underline{r}$ in the S-CIR specifications are very stable and nearly identical between the CRP and TVRP cases. In the S-CIR constrained model (Figure 4), the lower bound $\underline{r} \leq 0$ is always either at 0 or slightly negative, which allows estimating the model even when interest rates are slightly negative. In the unconstrained case (Figure 5), the lower bound $\underline{r}$ is well above zero in both markets for sub-samples ending before 2007, reaching values as high as 3% in some cases. Beyond the fact that the S-CIR specification allows to estimate the model even under mildly negative interest rates, the unconstrained specification improves the fitting of the conditional interest rate volatility by estimating the level at which the instantaneous volatility of the interest rate vanishes. In the original CIR model, as well as in the constrained S-CIR for most sub-samples, this level is equal to 0. In the following paragraphs, it will be clear whether this additional flexibility allows improving out-of-sample forecasts.

2.7. Out-of-sample forecasting ability

Panels (c) and (d) of Figure 1 show the ratio, for a 6-month forecasting horizon, between the RMSE of the out-of-sample yield forecasts and the RMSE under the random walk hypothesis. Two results are very strong and common to both datasets. First, the TVRP variants of all models have weaker forecasting ability than their CRP counterparts. This is especially true for the Vasicek model. Second, all unconstrained specifications of the S-CIR model outperform the corresponding constrained ones. For the U.S., the Vasicek and the S-CIR CRP unconstrained model are those with the best forecasting ability, being the best performers respectively at the short and the long end of the yield curve. For Germany, the S-CIR CRP unconstrained model is by far the best performer.

2.8. Pre-2008 out-of-sample forecasting ability

It is legitimate to wonder whether the evidence on the forecasting ability of the different models is different when considering periods when interest rates are either far above zero or always close to the zero level. In order to address this issue, the out-of-sample analysis is repeated over two different sub-samples, one ending before the 2008 financial crisis, i.e., in December 2007, and another one starting from January 2008. Panels (a) and (b) of Figure 6 refer, respectively, for U.S. and Germany to the pre-2008 period. It is important pointing out that over this first sub-sample, the constrained S-CIR model coincides with a regular CIR model, as the estimate of the lower bound $\underline{r}$ is equal to 0 at all times (see the time series of the estimates in Figure 4). For the unconstrained S-CIR model, instead, $\underline{r}$ is strictly positive over this sub-sample. For both datasets, a common result is that models with constant risk premia have a rather similar forecasting ability and outperform the corresponding models with time-varying risk premia. The Vasicek model is the best performer for the U.S. dataset, followed by the unconstrained S-CIR model, which has similar performances for long maturities and then by the constrained S-CIR model. For what concerns the German dataset, for short maturities, the best performer is the constrained Shifted-CIR model, followed by the Vasicek and the unconstrained S-CIR model. For long maturities, the forecasting abilities of the specifications with constant risk premia are indistinguishable.

2.9. Post-2008 out-of-sample forecasting ability

Panels (c) and (d) of Figure 6 refer to the post-2008 period when interest rates have been close to the zero level most of the time. For the U.S. dataset, the S-CIR specifications have better forecasting ability than the Vasicek specification, with a slight edge for the unconstrained S-CIR model. Interestingly, while allowing for a time variation of the risk premia is strongly detrimental for the Vasicek model, it provides a slight benefit to the S-CIR models. For the German dataset, the unconstrained S-CIR model is again the best performer, followed by the constrained S-CIR and

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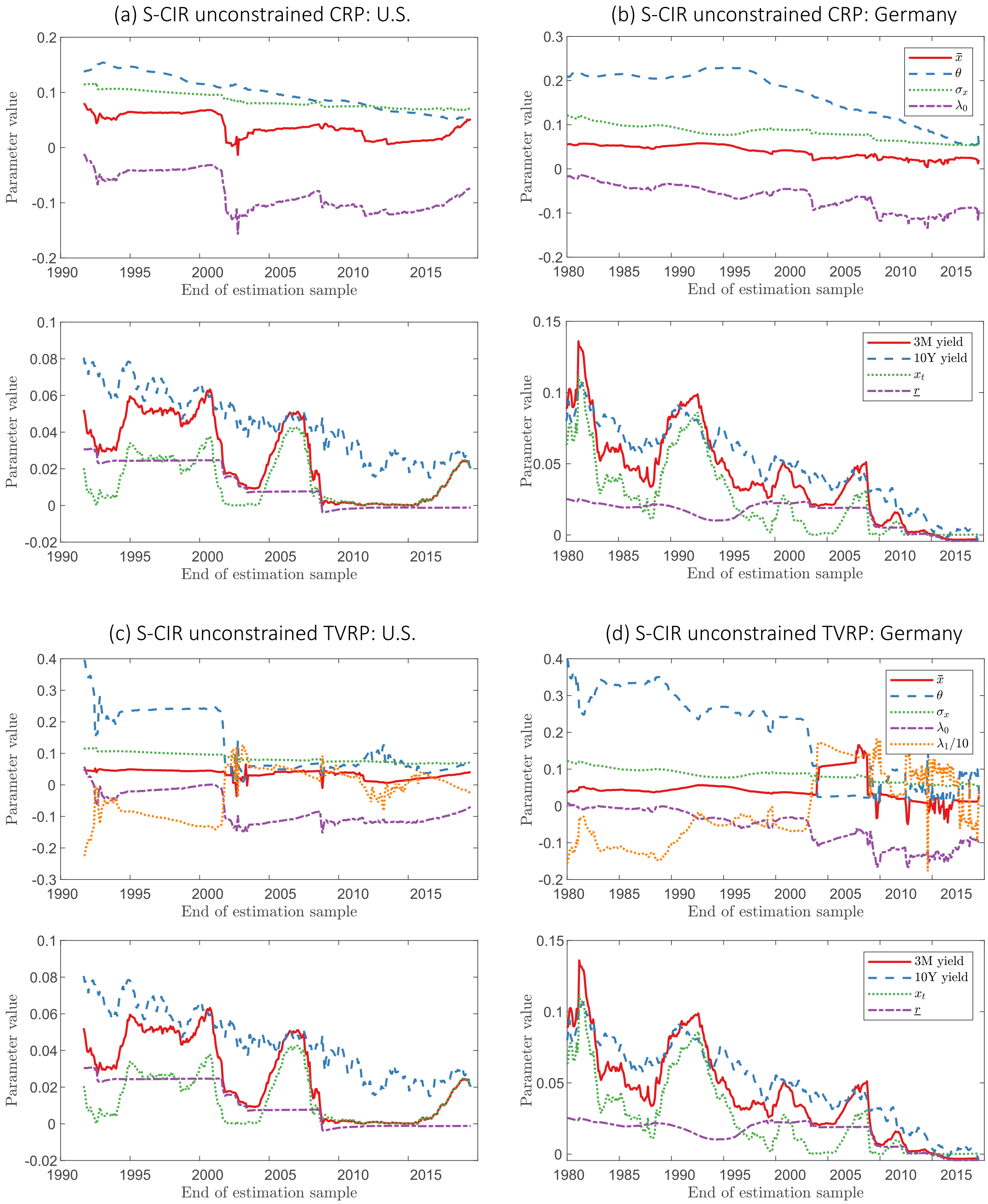

Note: The figure shows out-of-sample parameter estimates obtained for the S-CIR specifications with constant (CRP) and time-varying risk premia (TVRP). The lower bound to the interest rate $\underline{r}$ is not constrained. The initial estimation window is made of 240 monthly observations, starting on August 1971 for the U.S. dataset and on January 1960 for the German dataset. The X-axis values represent the end of the estimation window. x_t represents the end-of-estimation-period value of the unobservable state variable.

Figure 5. Out-of-sample parameter estimates for the S-CIR models with an unconstrained lower bound

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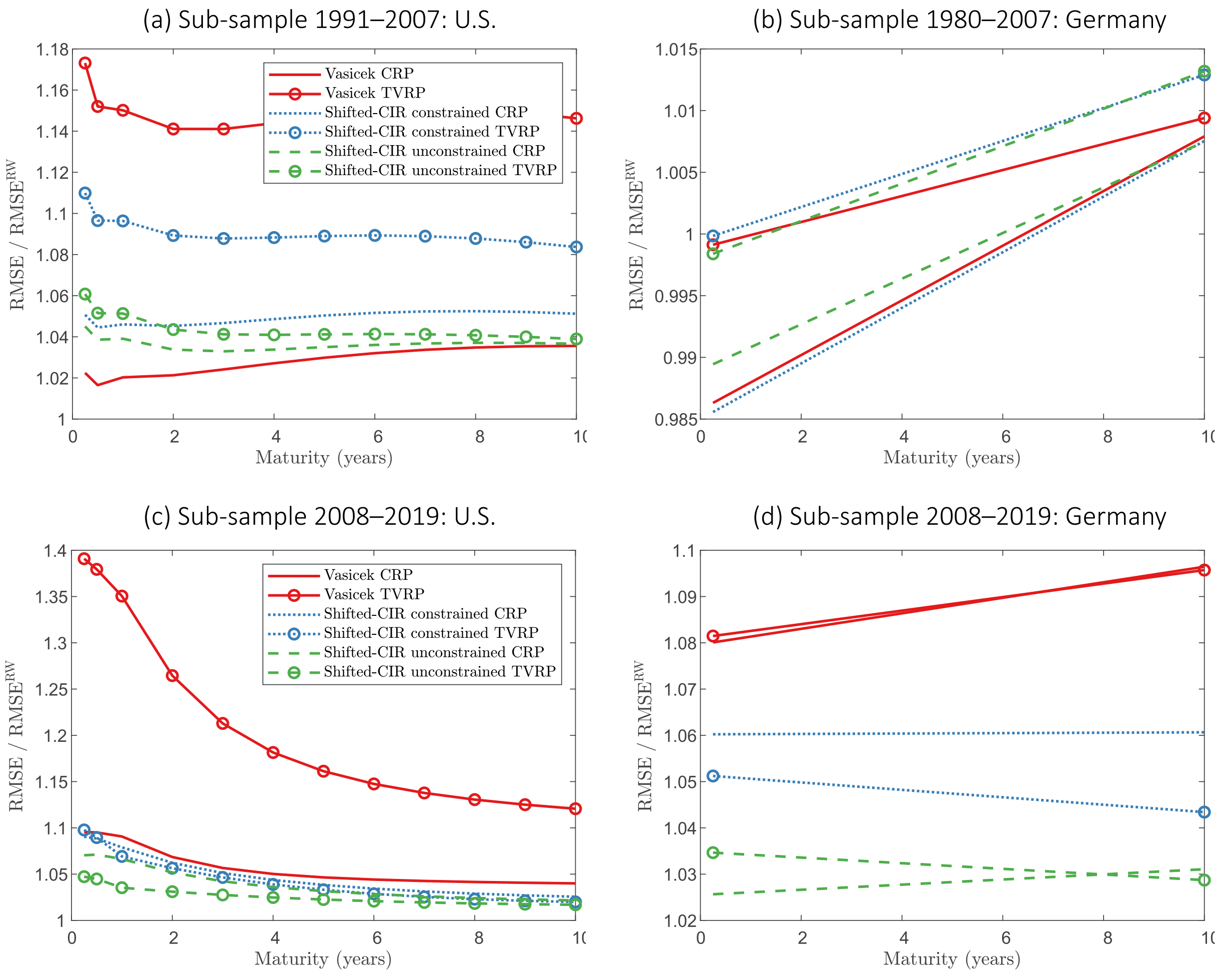

Note: For the different maturities considered (3 months to 10 years), the figure shows the ratio between the root-mean-squared errors of yield variations forecast obtained with the different models considered (RMSE) and the root-mean-squared errors of yield variations obtained under the random walk hypothesis $\left(\mathrm{RMSE}^{\mathrm{RW}}\right)$. The forecasting horizon is $\Delta t = 6$ months. For the U.S. dataset, the analysis is performed over two out-of-sample sub-samples respectively ranging from August 1991 to December 2007 and from January 2008 to June 2019. For the German dataset, the analysis is performed over two out-of-sample sub-samples, respectively, ranging from January 1980 to December 2007 and from January 2008 to July 2019.

Figure 6. Out-of-sample 6-month-ahead yield variation forecasting ability in comparison to the random walk hypothesis (sub-samples pre-2008 and post-2008)

the Vasicek model. Allowing for time-varying risk premia improves the performance only for the constrained S-CIR.

2.10. Out-of-sample volatility forecasting

Panels (c) and (d) of Figure 2 show the out-of-sample volatility forecasts as opposed to the 3-month rate realized volatility. The picture is similar to the in-sample analysis, where the Vasicek model, except than for the decreasing trend due to the variation of the out-of-sample estimates for σ_x, cannot predict the time variation of the interest rate volatility. Conversely, the S-CIR specifications explain a significant part of the empirical variability, with an R^2 of 51.5% (constrained) and 50.7% (unconstrained) for the U.S. and of 47.2% (constrained) and 49.9% (unconstrained) for Germany.

3. DISCUSSION

Under the assumption of constant risk premia, both considering U.S. and German data, the S-CIR specification is better than the Vasicek model in terms of information criteria. It also better matches the volatility of interest rates, both unconditionally and

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conditionally, especially for the U.S. sample. The in-sample forecasting ability is similar between the Vasicek and S-CIR specification, with a slight edge in favor of the first for short maturities and of the second for long maturities. In the out-of-sample analysis, the forecasting ability of the S-CIR specification is also comparable to or stronger than that of the Vasicek specification in most cases, again with better performances over long maturities. The out-of-sample findings also suggest that, as opposed to a regular CIR model, the additional flexibility of the S-CIR model, allowing to define a nonzero interest rate level at which the volatility vanishes, can be beneficial even when the interest rates are far above zero. Finally, according to nearly all the above metrics and criteria considered, introducing a time variation of the risk premia is harmful, especially when considering a Vasicek specification.

CONCLUSION

This paper empirically studies a Shifted-CIR (S-CIR) specification for the no-arbitrage pricing of the yield curve as a tractable one-factor alternative to the Vasicek process. This specification is particularly interesting if interest rates can assume mildly negative values when a traditional zero-lower-bound CIR model is undefined.

For both Vasicek and S-CIR models, the empirical analysis recommends assuming a constant market price of interest rate risk. Indeed, introducing a time-varying risk premium reduces the significance of the estimates, the model quality in terms of AIC and BIC criteria, as well as the in- and out-of-sample forecasting ability. These detrimental effects are particularly strong for the Vasicek model.

The information criteria identify the S-CIR-type models as preferred to Vasicek-type ones. The two types of models have similar in-sample yield forecasting ability, while in out-of-sample, there seems to be a slight edge in favor of the S-CIR model when the lower bound is not constrained and thus allowed to be positive. The S-CIR specification also entails a conditional time-varying interest rate volatility, which explains a significant fraction of the empirically-observed time variation of interest rate volatilities.

AUTHOR CONTRIBUTIONS

Conceptualization: Andrea Tarelli.
Data curation: Andrea Tarelli.
Formal analysis: Andrea Tarelli .
Investigation: Andrea Tarelli.
Methodology: Andrea Tarelli.
Project administration: Andrea Tarelli.
Software: Andrea Tarelli.
Supervision: Andrea Tarelli.
Validation: Andrea Tarelli.
Visualization: Andrea Tarelli.
Writing – original draft: Andrea Tarelli.
Writing – review & editing: Andrea Tarelli.

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