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Ruin probability: a flexible approach for measuring portfolio risk

Abstract

The paper provides a simple and flexible numerical based approach to measure portfolio risk. It measures the chance of one or more specific undesirable financial events, defined as Ruins, for a predetermined investment horizon. The method is based on the Ruin theory and is flexible enough to provide information about several financial events of concern at the same time and still relate them. The authors also conduct a simulation analysis using Bootstrap and Monte Carlo methods, in order to exemplify how this approach is able to solve some practical risk management problems.

Keywords: risk measures, portfoloio risk, Ruin probability.

Introduction

Several researchers tried to understand what exactly risk means and how to control it efficiently in financial markets. Wood Jr. (1964) proposed that risk may be defined as either the chance or the uncertainty of a loss. Later, the word "risk" received new connotations, such as changes in values between two dates. However, Artzner et al. (1999) argued that because risk is related to the variability of the expected value of a position, it is better to relate risk only with the future value of that position. Therefore, an exact and also comprehensive definition of risk leaves us with the notion that this concept is too broad to be represented by only one definition.

For instance, in the context of portfolio risk management, Markowitz (1952) was the first who formally define risk as the standard deviation of portfolio's returns. Several other studies came up with new measures of portfolio risk. Some of the most used portfolio risk measures in the financial market are the Beta[1], defined by the capital asset pricing model (CAPM), the implied volatility from the Black-Scholes (B-S) model[2], the portfolio Value at Risk[3] (VaR) and the portfolio expected shortfall[4].

Regardless of the existence of alternative risk definitions and measures, they are not mutually exclusive. In other words, the existence of one risk measure does not invalidate the usage of others as complementary risk measures for a specific investment. Actually, the risk management industry uses several alternative models to generate a higher number of measures in order to efficiently access and control portfolio risk.

Actuarial risk measures are related to financial risk measures in the sense that both try to measure the risk of loss in a portfolio. Furthermore, Shyriaev et al. (1999) argue that it is impossible nowadays to consider the theory and practice of insurance separated from the practices and the theory of finance and investment in securities. This leads us to believe that actuarial methodologies may be useful in measuring asset portfolio risk.

In this perspective, the main objective of this article is to introduce a flexible approach for measuring asset portfolio risks based on an actuarial concept, the Ruin probability[5]. The risk measure, we propose here, is also defined as the Ruin probability and it measures the chance of occurrence for one or more specific undesirable financial events related to portfolio's future value, defined as Ruins, for a predetermined investment horizon.

We create neither a new numerical simulation approach nor a new backtest methodology. Instead, we applied a widely used concept of risk for insurance surplus (Ruin probability) to asset portfolio risk management. By doing so, we make possible to specify as many events of concern as desired and evaluate the probabilities of these events, as independent risk events and also as correlated events, in an alternative approach to the widely used statistical measures of standard deviation and semivariance, underlying the classical measures of risk mentioned above.

In addition to the definition of risk, related to the future value of a position, Artzner et al. (1999) also presented four desirable properties[6] for measures of risk and call the measures satisfying theses properties as "coherent measures of risk". Nevertheless, the definition of risk, used here to develop the Ruin probability, is associated with the idea of a probability of a specific event of concern (a severe loss, for example) and, therefore, is related to the concept of risk presented by Wood Jr. (1964). As a conse-

[1] Portfolio (or asset) sensibility to the market. The CAPM was developed by Sharpe (1964).

[2] The Black-Scholes model was first introduced by Black and Scholes (1973) and then improved by Merton (1973).

[3] Portfolio maximum loss expected for a certain significance level.

[4] This is the portfolio expected loss conditional on a specific VaR.

[5] This concept of risk is used by actuaries in the context of insurance companies. The Ruin probability, in this scenario, is the probability that the insurer's surplus become negative at some point of time (for a defined or infinite horizon). The concept is developed by the Ruin theory, which is concerned with the level of an insurer's surplus for a portfolio of insurance policies. As more detailed references, see Kaas et al. (2001), Dickson (2005), and Boland (2007).

[6] They are: translational invariance, subadditivity, positive homogeneity and monotonicity.

quence, our measure does not necessarily satisfies the four properties proposed by Artzner (1999), since the definition of risk here used is different than the proposed by his work. Measures of probability of events, such as the Ruin probability, might present the second and the last desirable properties (subadditivity and monotonicity) of a "coherent measure of risk", but cannot present the other ones (translational invariance and positive homogeneity)[1].

This paper is structured in five more Sections. Section 1 contains a description of the theoretical background, which must be used to evaluate the Ruin probability. Section 2 presents the hypothetical situation that works as our motivation for the simulated analysis presented in this paper. Following, Section 3 gives an overall description of the data collected and briefly explains the numerical simulation methodologies that were used for the simulation analysis. Section 4 presents the results of the practical analyses and it is divided into the backtest for the numerical approaches and the final results. Finally, the last Section presents some conclusions about the possibilities and relevance of our method.

1. Theoretical background

Consider a portfolio of n assets during the future investment horizon $[0, H]$. The portfolio's value at time t is a random variable defined as $P_t \ \forall \ t = [0, H]$ with a probability density function $f(P_t)$ which also depends on t. Still, consider a portfolio risk manager, who is concerned about N possible undesirable events, related to the portfolio's value during the investment horizon, which are defined as Ruins. As a consequence, the Ruins add risk to the portfolio and must be analyzed.

Then, define N dependent Bernoulli random variables conditioned on the probability density function for the portfolio's value during the investment horizon, $R_i[f(P_t)]$, $i = 1,2,\dots,N$ and $t = [0, H]$, which represent the possible Ruins for the portfolio:

$$R_i\left[f(P_t)\right] = \begin{cases} 1, \text{if the } i-th \text{ Ruin occurs for the investment horizon } [0, H] \\ 0, \text{ if the } i-th \text{ Ruin does not occur for the investment horizon } [0, H]. \end{cases} \tag{1}$$

The risk manager must be aware the probability of occurrence for each Ruin. This implies that he must somehow evaluate $P[R_i = 1 | f(P_t)] \ \forall \ i = 1,2,\dots, N$ and $t = [0, H]$, which are defined as the Ruin probabilities[2]. If $f(P_t) \ \forall \ t = [0, H]$ is known, then it is possible to calculate the Ruin probabilities. Unfortunately, as $f(P_t) \ \forall \ t = [0, H]$ represent several future probability density functions of a complex portfolio, this is not the case.

Now, consider that the portfolio's value is divided into n assets such that the portfolio's total value is the sum of the assets' values:

$$P_t = \sum_{i=1}^{n} S_{i,t}, \tag{2}$$

where $S_{i,t}$ is the value of asset i at time t. As a consequence, the probability density functions of P_t depends on the sum of the values of the assets that compose the portfolio:

$$f(P_t) = f\left[P_t \middle| \sum_{i=1}^{n} S_{i,t}\right], \tag{3}$$

where $\varepsilon_{i,t}$, $i = 1,2,\dots, N$ and $t = [0, H]$ are independent random variables.

In order to estimate $P[R_i = 1 | f(P_t)]$, assumptions about $f(P_t)$ must be made or, otherwise, assumptions about changes in each $S_{i,t}$. Nonetheless, portfolios are composed by many different kinds of assets and, thus, no model is likely to work for all types of assets. For instance, we assume that all assets in the portfolio are public traded stocks[3].

Then we used a widely accepted multiplicative model for changes in stock prices such that:

$$\ln\left(S_{i,t+\Delta t}\right) = \ln\left(S_{i,t}\right) + \varepsilon_{i,t}, \tag{4}$$

where $\varepsilon_t \ \forall \ t = [0, H]$ are independent random variables.

If we assume a distribution for ε_t and a static or dynamic model for the parameters of the distribution, then we may simulate the process described in equation (4) for each portfolio's asset and estimate the future $f(P_t) \ \forall \ t = [0, H]$. Using the estimated $f(P_t)$, it is possible to calculate $P[R_i = 1 | f(P_t)] \ \forall \ i = 1,2,\dots, N$ and $t = [0, H]$. Furthermore, it is also possible to evaluate $P[E | f(P_t)] \ \forall \ t = [0, H]$, where E represents any event of concernment that relates two or more Ruins[4]. By doing this, the risk manager is able to estimate the Ruin probability for each specific event of his concern.

2. An hypothetical analytical situation

Let's assume that at the beginning of September 2010 a blue-chip fund is considering to create an equally weighted portfolio, composed by the preferred stocks of the 5 Brazilian companies with the greatest market

[1] The proof for this statement is presented in the Appendix.

[2] The Ruin probability of r_i is $P[R_i = 1 | f(P_t)]$.

[3] The model may be generalized for portfolios, composed by different types of assets such as derivatives, interest rates and commodities by making different assumptions about changes in prices of alternative assets. We rely on a unique type of asset in order to better explain the model's idea.

[4] Event such as "$R_i = 1 \cup R_j = 1$", "$R_i = 1 \cap R_j = 1$" or "$R_i = 1 | R_j = 1$".

capitalization listed in the BM&FBovespa[1]. The manager will rebalance the portfolio after one year and, therefore, he is concerned about the occurrence of two specific undesirable events (Ruin I and Ruin II) for the investment horizon of one year:

- Ruin I – the portfolio value decreases by k% before the end of the investment horizon;
- Ruin II – the portfolio value do not increase by the end of the investment horizon.

The simulated analysis of this paper is constructed to solve the problem presented in this hypothetical situation (to measure the risk of both Ruin I and Ruin II). Despite of the simplicity of the portfolio structure, the analysis intends to illustrate how the approach for measuring the Ruin probability is flexible enough to deal with different risk events as separated or dependent risk factors.

3. Data structure and numerical simulation methods

3.1. Data structure. For the simulated analysis, data on preferred stocks of the 5 top Brazilian companies, ranked by market capitalization[2], were collected from the Economatica[3] database for the sample period ranging from January 1, 2000 to August 31, 2010. The data represents the adjusted daily opening price such that for each stock the returns were calculated considering the difference in the natural logarithm of the opening prices of two subsequent market days. The whole analysis was performed using two different softwares, Microsoft Excel 2007 and the R statistical package.

3.2. Numerical simulation methods. To evaluate the Ruin probability for the hypothetical situation described in the Section 2, it was necessary to perform simulations of the portfolio returns for the following year. In order to do so, three alternative simulation approaches were considered for the stocks that composed the portfolio.

3.2.1. Bootstrap simulation. The Bootstrap is a data-based non-parametric simulation method and the details for the methodology can be found in Efron and Tibshirani (1993). For our analysis, we used the equation (4) for each stock and simulated future values of $\varepsilon_{i,t}$ using the data collected and considering that all observed returns had the same probability of happening.

3.2.2. Unconditional Monte Carlo simulation. We performed a Monte Carlo simulation considering that the $\varepsilon_{i,t}$ in equation (4) are independent and normally distributed with constant parameters for the normal distribution (μ and σ). In other words, we assumed a geometric Brownian motion[4] (GBM) for the price of each stock and we consider that the parameters of the GBM are constant and unconditional over time. Then, recovering the definition of equation (4), with a slight difference for the error term such that:

$$\ln\left(S_{i,t+\Delta t}\right) = \ln\left(S_{i,t}\right) + u_{i,t}, \tag{5}$$

where $u_{i,t} \sim N(\mu_i \cdot \Delta t\ ,\ \sigma_i \cdot \sqrt{\Delta t})$ and are i.i.d. for the same stock i.

The parameters μ_i and σ_i are respectively the mean and the standard deviation of the observed returns[5] of the stock i. We also considered the cross-section correlation structure of the portfolio assets[6].

3.2.3. Conditional Monte Carlo simulation. This is also a Monte Carlo simulation considering a GBM for the price of each stock and the cross-section correlation structure of the portfolio assets. Nevertheless, the σ is considered to be conditional over time. That is, $\sigma_{i,t}$ depends on the stock i and time t. The conditional process considered is the generalized autoregressive conditional heteroskedasticity (GARCH) model, proposed by Bollerslev (1986). We used a GARCH (1,1), such that for each stock we have:

$$\sigma_{i,t}^2 = \alpha_0 + \alpha_1 r_{i,t-1}^2 + \alpha_2 \sigma_{i,t}^2, \tag{6}$$

where $r_{i,t}$ is the return of stock i at time t.

3.2.4. Further issues. We do not pursue general or further descriptions of the numerical approaches, but the general methodology for simulating portfolios' returns, using these approaches, can be found in Jorion (2001) and Christoffersen (2003).

Also, there are three arbitrary general rules chosen to implement the simulations:

- the number of trading (market) days in one year and one month are, respectively, 252 and 21 days;
- for each simulation, a DataWindow of K years was selected[7] and the respective database for the DataWindow chosen was used to proce-

[1] This is the Brazilian stock exchange. More information about it can be obtained at http://www.bmfbovespa.com.br/en-us/home.aspx?idioma=en-us.

[2] The top five market capitalization Brazilian companies was obtained at the official website of the BM&FBovespa and the top five companies were, respectively: Petróleo Brasileiro S.A (Petrobras), Vale S.A, Itaú Unibanco Banco Múltiplo S.A, Companhia de Bebidas das Americas (AmBev) and Banco Bradesco S.A.

[3] Economatica is a financial database that contains a wide coverage of the Brazilian stock market.

[4] The Brownian motion is the limit of the random walk model, first proposed by Bachelier (1900), translated to English by Cootner (1964), and the geometric Brownian motion was proposed by Samuelson (1965) as an adjustment of the linear Brownian motion to consider some properties of stock prices.

[5] $r_{i,t} = ln(S_{i,t+1}) - ln(S_{i,t})$.

[6] In order to consider the cross section correlation structure of the portfolio assets we used the Cholesky decomposition. Details of this method can be found in Jorion (2001).

[7] The DataWindow represents the number of years of sample that are going to be used for the simulation approach. We used two alternative DataWindows, 5 years and 10 years.

dure the simulation approach. Furthermore, for each simulation the database used contained $K{\cdot}252$ daily returns for five different stocks, with the following general structure for the database:

$$\text{Database}=\begin{bmatrix} r_{1,1} & r_{1,2} & r_{1,3} & r_{1,4} & r_{1,5} \\ r_{2,1} & r_{2,2} & r_{2,3} & r_{2,4} & r_{2,5} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ r_{K\cdot 252,1} & r_{K\cdot 252,2} & r_{K\cdot 252,3} & r_{K\cdot 252,4} & r_{K\cdot 252,5} \end{bmatrix},$$

where $r_{i,j}$ represents the return of stock i at day j;

- for each simulation approach and DataWindow chosen, 50,000 simulations were run.

4. Results

In order to evaluate the probabilities of Ruin I and Ruin II it is necessary to verify if the numerical models, used to implement the simulations, were accurate for approximating the distribution of the portfolio's returns. Therefore, before choosing the model, we performed backtests for all of the three proposed simulation approaches. We did that by using the last five months of data[1] for the backtests.

4.1. Backtesting procedure. There are several alternative backtesting procedures to deal with problems like these. Kupiec (1995) argued that formal statistical procedures relying in occurrence of events of low probability[2] require large samples to produce a reliable assessment of a model's accuracy. Additionally, as we are usually concerned about several undesirable events for the implementation of the Ruin probabilities, it is necessary to use a backtest which allows analyzing the performance of the simulation method for the entire distribution. We deal with these problems by using the widely accepted method, proposed by Berkowitz (2001) mainly because:

- it backtests the entire distribution instead of just the event of interest and, as a consequence, it enables the backtest even when the sample used for the backtest is restricted;
- according to tests performed by Berkowitz (2001), it is the most powerful methodology for backtesting a simulated distribution of returns.

As the main propose of our study is neither to evaluate different simulation approaches nor to analyze backtesting methodologies, we presented a brief description of the backtest performed.

To implement the backtests, we, first, computed the empirical cumulative distribution function (ECDF) of the portfolio daily returns for each simulation approach[3] and DataWindow. Then, for each backtest we used the ECDF to calculate the cumulative probability of each actual daily return of the backtest sample. This gives a vector of probabilities, which Rosenblatt (1952) shown, under the assumption (null hypothesis), that the simulated distribution is the right distribution, the probabilities must be independent and uniformly distributed the [0,1] range. Then, we used the inverse of the standard normal distribution function to transform the probabilities. Finally, according to Berkowitz (2001), under the null, the transformed probabilities must be independent and follow a standard normal distribution.

To perform the test in order to verify if the transformed probabilities were independent and followed a standard normal distribution, we used the following equation:

$$z_t - \pi = \rho \cdot \left(z_{t-1} - \pi\right) + w_t, \tag{5}$$

where z_t is the transformed probability at time t, π is the mean of the transformed probability, ρ is the first order autoregressive coefficient and w_t is the error of the model.

Then, we defined the log-likelihood as a function of the three unknown parameters, $L(\pi,\partial^2, \rho)$, where ∂^2 is the variance of the error. The null implies that $\pi = 0$, $\rho = 0$ and $\partial^2 = 1$. Therefore, we used the likelihood ratio (LR) test to test the null hypothesis. The likelihood ratio test statistic was defined as:

$$LR = -2 \cdot \left(L(0,1,0) - L\left(\hat{\mu}, \hat{\partial}^2, \hat{\rho}\right)\right). \tag{6}$$

Under the null hypothesis, the test statistic follows a $\chi^2(3)$.

Table 1 presents the p-values of the likelihood ratio tests for each simulation approach and DataWindow considered. The results indicated that the conditional Monte Carlo simulation approach was the only one that provided a reliable forecasted distribution for the backtest period, using a significance level of 10%. It is also noticeable that the 10 years DataWindow provides a better sample comparing to 5 years DataWindow in order to perform the simulations for all the three simulation methods considered. In summary, the backtesting results indicate that the best simulation approach is to use the conditional Monte Carlo simulation with a DataWindow of 10 years, under the assumptions used in our study.

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[1] The backtest period ranged from April 1, 2010 to August 31, 2010.

[2] As the Ruin can be defined as any financial event of concernment, this might be the case of our approach in some situations.

[3] For the conditional Monte Carlo simulation there is more than one ECDF because the ECDF is a function of the volatility and according to this method the volatility is conditional. Nevertheless, the idea for the backtes is similar, but there is one ECDF for each day of backtest.

Table 1. Simulation approaches backtest

DataWindow	Simulation approach		
	Bootstrap	Unconditional Monte Carlo	Conditional Monte Carlo
5 years	0,0050	0,0000	0,2431
10 years	0,0729	0,0009	0,4611

Note: The tests are performed using the specification $z_t - \pi = \rho\cdot(z_{t-1} - \pi) + w_t$, where z_t is the transformed probability at time t, π is the mean of the transformed probability, ρ is the first order autoregressive coefficient and w_t is the error of the model. The null hypothesis states that $\pi = 0$, $\rho = 0$ and $\partial^2 = 1$.

The backtest also indicates that the Bootstrap simulation using the DataWindow of 10 years is an appropriate approach at a 5% significance level (but not at 10%) and that all other simulation approaches, here considered, are not statistically meaningful.

4.2. Ruin probabilities. The final simulations are performed using both the conditional Monte Carlo and the Bootstrap simulation. The backtesting indicated that the conditional Monte Carlo simulation generated the most accurate forecasted distribution for daily returns. However, the bootstrap results are also presented to exemplify a possible bias in the Ruin probability estimation due to the utilization of the wrong simulation model.

In Section 2, we defined Ruin I and Ruin II as the events of concern. Although, the probabilities of Ruin I and Ruin II are interesting to understand the risk of each event, it is still necessary to quantify the relation between them. In order to do so, we also analyzed the event “Ruin I ∩ Ruin II”. Moreover, using the three events already defined, we also estimated the probabilities of two other events of interest: “Ruin I ∪ Ruin II” and “Ruin II | Ruin I”. We did that relying on two basic probability rules:

- $P(Ruin\ I \cup Ruin\ II) = P(Ruin\ I) + P(Ruin\ II) - P(Ruin\ I \cap Ruin\ II)$;
- $P(Ruin\ II | Ruin\ I) = \dfrac{P(Ruin\ I \cup Ruin\ II)}{P(Ruin\ I)}$.

Table 2 provides the Ruin probabilities at September 1, 2010 for all the five events of interest for both DataWindows and the simulation approaches considered (Bootstrap and conditional Monte Carlo). The results indicate that (for the specific situation of the analyses) the Bootstrap simulation method overestimates the Ruin probabilities of Ruin I and Ruin II. It is also noticeable that the simulations using the DataWindow of 5 years generate higher Ruin probabilities. This is due to the financial crises occurred between 2008 and 2009 (stocks’ returns for this period gains special weight as we use smaller DataWindow).

The analysis of the Ruin probabilities of Table 2 is not interesting by itself, since there is no special concern in the equally weighted portfolio of the five biggest Brazilian blue chips (it is just a hypothetical portfolio to exemplify the method). On the other hand, it is outstanding how the proposed methodology was able to relate all risks of interest in a couple probability measures.

For instance, by analyzing the results of the best simulation approach (conditional Monte Carlo simulation using a DataWindow of 10 years), the blue chip fund manager is able to verify that for the proposed portfolio the Ruin probability of a loss higher than 20% before the end of one year is as high as 15,69%. Nevertheless, the probability of a loss higher than 30% is much lower (4,72%). Also, the probability of no gain after one year is 18,35%. These results may determine whether or not the manager is going to invest in such a portfolio.

Table 2. Estimated Ruin probabilities

Panel A. Bootstrap simulation						
DataWindow	k%	Ruin probabilities				
		Ruin I	Ruin II	Ruin I ∪ Ruin II	Ruin I ∩ Ruin II	Ruin II | Ruin I
5 years	10%	53,84%	25,01%	54,79%	24,07%	44,70%
	20%	28,75%	25,01%	34,73%	19,03%	66,19%
	30%	13,00%	25,01%	26,92%	11,10%	85,36%
	40%	4,44%	25,01%	25,19%	4,27%	96,13%
	50%	1,06%	25,01%	25,01%	1,06%	100,00%
10 years	10%	44,86%	19,97%	46,18%	18,66%	41,59%
	20%	19,77%	19,97%	26,35%	13,39%	67,73%
	30%	6,98%	19,97%	20,82%	6,13%	87,84%
	40%	1,76%	19,97%	20,01%	1,72%	97,84%
	50%	0,27%	19,97%	19,97%	0,27%	99,25%

Table 2 (cont.). Estimated Ruin probabilities

Panel B. Conditional Monte Carlo simulation						
DataWindow	k%	Ruin probabilities				
		Ruin I	Ruin II	Ruin I $\cup$ Ruin II	Ruin I $\cap$ Ruin II	Ruin II | Ruin I
5 years	10%	48,07%	22,74%	49,57%	21,24%	44,19%
	20%	21,67%	22,74%	29,35%	15,05%	69,47%
	30%	8,16%	22,74%	23,73%	7,17%	87,92%
	40%	2,28%	22,74%	22,82%	2,21%	96,67%
	50%	0,43%	22,74%	22,75%	0,43%	98,16%
10 years	10%	40,73%	18,35%	42,21%	16,86%	41,40%
	20%	15,69%	18,35%	23,06%	10,98%	69,98%
	30%	4,72%	18,35%	18,85%	4,22%	89,44%
	40%	0,99%	18,35%	18,38%	0,96%	96,96%
	50%	0,12%	18,35%	18,35%	0,12%	100,00%

Notes: The Table presents the estimated Ruin probabilities for the five events of interest: “Ruin I”, “Ruin II”, “Ruin I $\cup$ Ruin II”, “Ruin I $\cap$ Ruin II”, “Ruin II | Ruin I”. The probabilities are estimated using two alternative simulation approaches (Bootstrap and Conditional Monte Carlo) and two Data Windows (5 years and 10 years). Ruin I is defined as the event in which the portfolio value decreases by k% before the end of the investment horizon and Ruin II is the event in which the portfolio value do not increase by the end of the investment horizon. We evaluate the probabilities using k% = 10%, 20%, 30%, 40% and 50% and the investment horizon of one year.

Conclusion

The existing portfolio risk measures are important for the risk management industry in order to efficiently access and control the portfolios risk. On the other hand, each measure is specific for one kind of risk. For instance, the Beta of the CAPM is a measure of the portfolio market sensibility and the value at risk is a measure of extreme high (and unlikely) losses.

Although, the existing risk measures are important in determining specific risk factors, they fail in providing flexible analysis of events in the sense that it is possible that the event of concern for the manager is not related to the available risk measures. The contribution of this paper is to provide a simple approach that deals with this issue. In doing so, with the adoption of the Ruin probability approach here proposed, the risk managers are able to specify in a comprehensive manner one or more events of interest and even to access their conditional and unconditional probabilities.

Despite its simplicity, the hypothetical example analyzed in this paper is a clear situation in which the traditional risk measures would fail in providing the risk manager with relevant information about his concernments. Furthermore, the Ruin probability was able to solve the proposed problem especially because of its flexibility.

Finally, the approach may be further investigated with real portfolios, which usually present a more complex assets structure. Also, the problems may be expanded to take into account assets purchase decisions. The manager may also specify a specific event of interest and analyze the change in the portfolio’s Ruin probability due to the purchase of the specific asset of interest. Finally, the results we present here motivate several real-world risk management applications which we leave for future research.

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Appendix

Consider two portfolios with values at time t defined respectively by X_t and Y_t, such that t might be a specific time or a period and that $X_0 = Y_0$. Still, assume that X_t and Y_t are random processes and that we are concerned about the event $Z_t < A$, where Z_t is the value of a generic portfolio at time t and A is a know value, which might be an absolute value or a fraction of Z_0. The Ruin probability is, therefore, $P(Z_t < A)$.

We shall prove that the Ruin probability presents two of the properties of a "coherent measure of risk" (subadditivity and monotonicity) and that it is impossible for the Ruin probability to present the other two properties (translational invariance and positive homogeneity), since it is a measure of probability.

1. Subadditivity.

Applied to the Ruin probability, subadditivity implies that the following inequation is true:

$$P(X_t + Y_t < A) \leq P(X_t < A) + P(Y_t < A).$$

The event "$X_t + Y_t < A$" implies that "$X_t < A \cap Y_t < A$", but the opposite is not true. Therefore, we have that:

$$P(X_t + Y_t < A) \leq P(X_t < A \cap Y_t < A)\ . \tag{A1}$$

Now, let's consider the following probability rule and apply Inequetion (A1) to it:

$$P(X_t < A \cap Y_t < A) = P(X_t < A) + P(Y_t < A) - P(X_t < A \cup Y_t < A),$$

$$P(X_t + Y_t < A) \leq P(X_t < A) + P(Y_t < A) - P(X_t < A \cup Y_t < A)\ . \tag{A2}$$

As $P(X_t < A \cup Y_t < A) \geq 0$, inequation (A2) also implies that:

$$P(X_t + Y_t < A) \leq P(X_t < A) + P(Y_t < A).$$

2. Monotonicity.

Applied to the Ruin probability, monotonicity implies that if $X_t \geq Y_t$ is true, then the following inequation is also true:

$$P(X_t < A) \leq P(Y_t < A).$$

Assuming that $X_t \geq Y_t$ is true, we have that if $X_t < A$ it is certain that $Y_t < A$:

$$P(Y_t < A | X_t < A) = \underbrace{\frac{P(Y_t < A \cap X_t < A)}{P(X_t < A)} = 1}_{(B1)}.$$

The result of equation (B1) is:

$$P(Y_t < A \cap X_t < A) = P(X_t < A)\ . \tag{B2}$$

Now, consider the following probability rule and apply equation (B2) to it:

$$P(Y_t < A \cup X_t < A) = P(X_t < A) + P(Y_t < A) - P(X_t < A \cap Y_t < A),$$

$$P(Y_t < A \cup X_t < A) = P(X_t < A) + P(Y_t < A) - P(X_t < A),$$

$$P(Y_t < A \cup X_t < A) = P(Y_t < A). \tag{B3}$$

Consider two events E and F. We know that the probability of occurrence for the event E is equal or less than the probability of occurrence for the event "$E \cup F$". Now, assume that E = "$X_t < A$" and that F = "$Y_t < A$". The result is:

$$P(X_t < A) \leq P(X_t < A \cup Y_t < A)\ . \tag{B4}$$

Then, substituting equation (B3) into inequation (B4) leaves:

$$P(X_t < A) \leq P(X_t < A).$$

3. Translational invariance.

Consider an initial amount of α_0 that is invested and its future value is determined by $\alpha_0 \cdot y$, where y is a positive (deterministic or random) value. The translational invariance applied to the Ruin probability states that:

$$P(X_t + \alpha_0 y < A) = P(X_t < A) - \alpha_0. \tag{C1}$$

Observe that $P\ (X_t < A)$ is a probability measure and that α_0 is an amount of money. As a consequence, if equation (C1) is true, then $P\ (X_t + \alpha_0 \cdot y < A)$ might present negative values or even values above 1 and this cannot be true.

4. Positive homogeneity

Positive homogeneity applied to the Ruin probability states that $P\ (\lambda \cdot X_t < A) = \lambda \cdot P\ (X_t < A)$ for all $\lambda \geq 1$. Observe that if positive homogeneity is true, then $P\ (\lambda \cdot X_t < A)$ might present values above 1 and as $P\ (\lambda \cdot X_t < A)$ is a probability measure, this cannot be true.