“Investment strategy performance under tracking error constraints”

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INVESTMENT STRATEGY PERFORMANCE UNDER TRACKING ERROR CONSTRAINTS

Abstract

Recent (2018) evidence identifies the increased need for active managers to facilitate the exploitation of investment opportunities found in inefficient markets. Typically, active portfolios are subject to tracking error (TE) constraints. The risk-return relationship of such constrained portfolios is described by an ellipse in mean-variance space, known as the constant TE frontier. Although previous work assessed the performance of active portfolio strategies on the efficient frontier, this article uses several performance indicators to evaluate the outperformance of six active portfolio strategies over the benchmark – subject to various TE constraints – on the constant TE frontier.

Keywords tracking error frontier, portfolio risk, active management, benchmarks

JEL Classification C52, G11

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INTRODUCTION

Passive managers follow a benchmark (index) tracking strategy, using financial instruments such as Exchange Traded Funds (ETFs) (Sharpe, 1991), whereas active managers seek investment opportunities with the objective of outperforming mandated benchmarks.

The competition between advocates of active and passive management has recently (2018) intensified. Historically, passive investing has taken preference globally, however, recent evidence identifies a change in this trend (Torr, 2018). The prevailing (2019) narrow bull market is prone to the fundamental weakness of being top-heavy: only six S&P 500 companies (namely Facebook, Apple, Amazon, Microsoft, Google and Johnson & Johnson) dominate the index tracking market. Passive managers have focussed on tracking the S&P 500, which has led to overcrowding of the ETF market (Brenchley, 2018). These high concentrations levels have reduced investors' only free lunch: diversification. Opportunity for real dominance by active managers would be marked by a reversal of this tide (Gilreath, 2017). The market risks of ETFs have been considerably underestimated, which led to – amongst others – the flash crash of August 2015 (Gilreath, 2017). This brief downtrend proved the falsehood of the assumption that intraday trading risks associated with ETFs are low (Gilreath, 2017).

Lambridis (2017) found the origin of the anti-active stance to be based on misperceptions stemming from the US experience. Almost all investment-style research to date (2019) has used US data, where market characteristics differ considerably from emerging markets. The inefficiency inherent in developing economies is not conducive to passive

management strategies. Emerging market investors do not enjoy the tax advantages provided by passive instruments based in the US (Lambridis, 2017). Moreover, the disproportionate weighting of South African (as an example of an emerging economy) indices lends itself to higher concentration risks, limiting diversification. Naspers Limited, for example, dominates South African indices (Cairns, 2018).

Torr (2018) warned that the increased concentration of market leadership itself is positive for active managers. Unlike US active funds, South Africa has seen a higher percentage of outperformance on an after-fees basis and South African indices have achieved some of the highest international turnover rates, a statistic that augurs badly for the pro-passive incentive of lower costs (Lambridis, 2017). The pursuit of active management strategies, therefore, has the potential for great reward.

Given the shift towards active management, the potential for this investment philosophy must first be supported by compelling evidence, proving its sustained performance advantage over passive strategies (Gilreath, 2017).

Modern Portfolio Theory (MPT) introduced Markowitz efficient portfolios, which are located on the efficient frontier, a curve in risk/return space representing utility maximizing investment opportunities above the minimum variance portfolio (Markowitz, 1952). Portfolio managers are only worth rewarding and retaining if their performance is, on average, positive (Roll, 1992). Since the debut of MPT, many measures have been developed to test the performance of portfolio strategies on the efficient frontier. Sharpe (1966) introduced the Sharpe ratio (returns greater than the risk-free rate over the total portfolio risk, σ to measure a fund manager's selection skill. The more popular Information Ratio (IR) measures the quotient of the expected active return and the tracking error (TE – the standard deviation of the difference between portfolio and benchmark returns). IR reveals the manager's ability and consistency to generate excess returns. Later, the more comprehensive and interpretable Modigliani risk-adjusted performance measure (L. Modigliani & F. Modigliani, 1997), also known as M^2 RAP, was developed to assess the risk-adjusted returns of a portfolio relative to its benchmark.

TE is in widespread use within the asset management industry as it is used as an indicator, with a given level of statistical reliability, to determine whether active managers add value over a benchmark (Roll, 1992), as well as limits the risk of performance fee incentivized managers (Jorion, 2003). TE does not provide information on the direction of return differences, as it only measures excess return volatility, thus, TE should not be used in isolation (Thomas, Rottschafer, & Zvingelis, 2013), but instead is best used in combination with other performance evaluators, such as the Information Ratio, and Sharpe ratio.

Optimizing fund returns subject to a TE is dependent on the investment mandate. External agents determine this mandate to prescribe targets for outperformance, risk-return profiles and investment strategies (Roll, 1992).

The industry maintains the widespread use of TE for risk control (Goodwin, 1998). Jorion (2003) recognized this emphasis and formalized the shape of constant TE portfolios, characterizing the locus of points in risk/return space corresponding to a TE constraint (i.e. portfolios with maximum return for a given level of risk and subject to a TE constraint). The relationship between expected return and variance for a fixed TE (the constant TE frontier) is described by an ellipse in mean-variance space (and a distorted ellipse in risk/return space). Jorion (2003) advocated portfolios with maximum return, subject to a tracking error and having the same risk as the benchmark.

Historically, the performance measures mentioned earlier have been used to test the performance of portfolio strategies on the efficient frontier. Such portfolio strategies were constructed in response to the problem of "optimality", which arose from investors' differing risk profiles and objectives. We introduce,

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for the first time, an evaluation of the portfolio performance of the six portfolio strategies (maximum return, minimum variance, Jorion's benchmark risk, maximum diversification, minimum intra-correlation and maximum Sharpe ratio) on the constant tracking error frontier. That is, we explore the behavior of investment strategies in terms of performance ratios subject to TE constraints.

The remainder of this paper proceeds as follows: a literature review in section 1 explores previous work on modern portfolio theory, TE frontiers, constant TE frontiers, the development of the six portfolio strategies and relative performance measures aimed at establishing value added. Section 2 provides the data and methodology adopted in this study, with reference to the mathematics applied. Section 3 discusses some insightful results and last section concludes.

1. LITERATURE REVIEW

1.1. Portfolio management

Modern Portfolio Theory (MPT) formulates the portfolio selection problem as a trade-off between risk and return (mean/variance) of a portfolio of assets (Markowitz, 1952). This model introduced the concept of efficient portfolios, which demonstrated the total risk-reducing effects of adding assets to investment portfolios, whilst maximizing investor utility (Chen, 2016). Such portfolios lie on a boundary known as the efficient frontier (Figure 1). Each point on this frontier represents (efficient) portfolios that generate the maximum return for given levels of risk in mean-variance space. It is important to note that points that do not lie on this frontier (i.e. to the right of the boundary) are inefficient.

Minimum variance portfolios have the lowest level of risk (variance) for given expected returns. The global minimum variance portfolio is the leftmost point of this frontier representing the lowest level of risk achievable (Markowitz, 1952). Given that investor preferences vary greatly, Markowitz (1952) indicated that all investor preferences are satisfied on the frontier.

The construction of the efficient frontier assumes no investment restrictions, but most portfolio managers face some form of mandated restrictions from external agents who constrain their asset allocation decisions. These external agents are not necessarily investment professionals, which leads to a benchmark that may be frequently inefficient. Restrictions may include limits (upper and lower) on asset class weightings (equities, bonds, etc), forbidding short selling and portfolio risk constraints in the form of TEs and β_s.

The MPT led to the establishment of Capital Market Theory (CMT), which introduced a risk-free (zero variance) asset, which has zero correlation with all risky assets and provides a risk-free rate of return (Treynor, 1961, 1999; Sharpe, 1964; Lintner, 1965; Mossin, 1966). Investors may thus add riskless assets to portfolios to reduce the total level of portfolio risk (Sharpe, 1964).

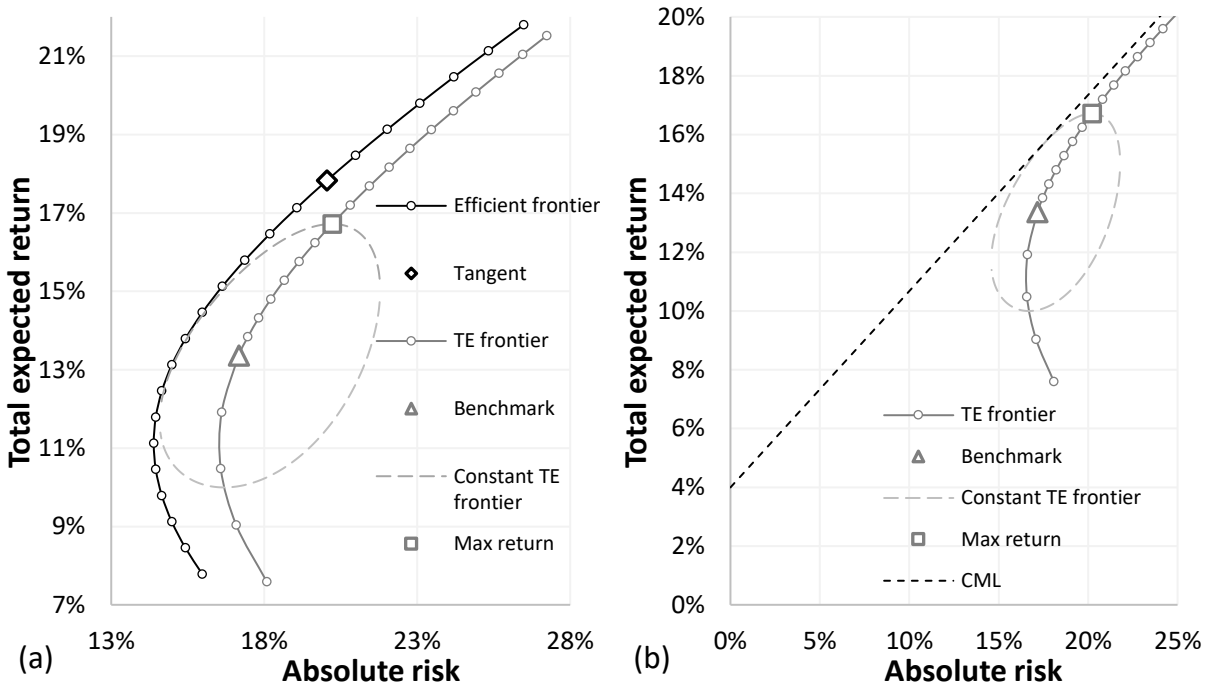
1.2. Performance measurements

Reilly and Brown (2009) asserted that the asset allocation decision is not an isolated choice but rather a cyclical process. First, an investment mandate is developed, reflecting the investor's risk profile and objectives. Then, current market conditions are analyzed to determine an appropriate asset allocation strategy, followed by a portfolio construction phase, in which financial theory is used to allocate funds across different industries and asset classes. The portfolio's performance is then continuously monitored and evaluated, as changes in the economy and investment mandate may require the portfolio manager to undertake corrective action (Reilly & Brown, 2009).

Rational investors are characterized by their preference for active or passive managers. Passive managers pursue a benchmark (usually a financial index) tracking strategy by holding securities from that benchmark (Sharpe, 1991), such as Exchange Traded Funds (ETF). Active managers are tasked with outperforming their benchmarks (usually determined by external agents) and are only deemed worth retaining if their performance is on average positive (Roll, 1992).

Active asset managers are assessed on the total return performance relative to a relevant bench-

Source: Roll (1992), Jorion (2003) and own calculations.



Note: Square marker indicates maximum Sharpe ratio on the global efficient frontier with no constraints. *TE* = 7% and $r_f = 4\%$.

Figure 1. (a) Efficient frontier, TE frontier and constant TE frontier and (b) efficient frontier, TE frontier and constant TE frontier and capital market line (CML) in mean/standard deviation space

mark (prescribed by in the investment mandate and which usually consists of a broadly diversified index of assets). These benchmark returns serve as an appropriate comparison with portfolio performance, because the benchmark provides a direct alternative to pursuing an active management strategy. Asset returns, however, have been shown to be exceedingly noisy (unexplained data variability) and thus long periods of evaluation are required to elapse before added value can be measured with statistical reliability (Roll, 1992). This lack of statistical reliability has led investors to direct their attention to – amongst other measures – the minimization of TE in the performance measurement of active managers (Roll, 1992). This performance strategy (deemed TE optimization) has two main objectives: outperformance of benchmark returns while simultaneously minimizing the TE.

In obtaining an additional constraint on the portfolio beta (β_p), Roll (1992) found that all TE constrained portfolios with positive expected performance will have a $\beta > 1.0$ (implying greater market risk to the benchmark) and resulting in a managed portfolio that does not dominate the benchmark. Roll (1992) thus proved the impossibility of constructing a portfolio that is simultaneously constrained by a TE, a given expected performance and a specified β, thereby demonstrating that minimizing TE does not result in more efficiently-managed portfolios. This was deemed the agency problem, which was further tested and confirmed by Jorion (2003).

Roll (1992) formulated the TE frontier in risk/return space. This curve represents the locus of maximal returns at given levels of portfolio risk for given levels of TE – the grey line in Figure 1(a). Jorion (2003) then formulated the assembly of the constant TE frontier, i.e. the locus of all returns at given portfolio risk levels, which correspond to a given TE constraint (not just the maximal return). This frontier is described by an ellipse (in traditional mean-variance space – the dashed grey line in Figure 1).

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Figure 1 shows the efficient frontier as a solid black curve. External agents set a prescribed benchmark in the investment mandate (grey triangle). Roll (1992) explains that it is uncommon for this benchmark to lie on the efficient frontier, as the external agent restrictions result in a benchmark that is often inefficient. Roll's (1992) TE frontier is also shown in Figure 1, as well as Jorion's (2003) constant TE frontier (in this example, TE = %7). Figure 1(b) shows the CML line for the constant TE frontier.

Given the shift towards active management, the potential for this investment philosophy must first be supported by compelling evidence, proving its sustained performance advantage over passive strategies (Gilreath, 2017). Performance measures serve as the vehicle for testing such portfolio strategies in pursuit of finding true value-added by portfolio managers.

Three performance metrics will be used in this article. Sharpe (1966) introduced the Sharpe ratio (a risk-adjusted return: the quotient of returns more than the risk-free rate and total portfolio risk, σ) to assess a manager's asset selection skill. The Information Ratio (IR) is the quotient of active return (returns greater than the benchmark) and TE. The IR reveals the manager's ability and consistency to generate excess returns. The Modigliani risk-adjusted performance measure (L. Modigliani & F. Modigliani, 1997), also known as M^2 RAP, measures the risk-adjusted returns of a portfolio relative to a benchmark.

1.3. Investment strategies

The problem of optimal performance arises as investors have varying risk profiles and objectives: the concept of an optimal portfolio is thus largely dependent on individual investor preferences. Several investment strategies have been constructed in response to the issue of optimality to allow investors to select their preferred utility maximizing strategy. Figure 2 illustrates the position of the six investment strategies that are investigated in this article on the constant TE frontier. Note that these positions move around the constant TE frontier as risk-free rates, benchmark weightings, asset choices, asset expected returns and asset covariances change. We have used a stylized set of these input parameters to demonstrate these movements. The six optimal portfolio investment strategies in common use include the maximum return, minimum variance, minimum intra-correlation, maximum diversification, maximum Sharpe ratio and Jorion's benchmark risk (following Jorion's (2003) definition of an optimal investment strategy, which involved aiming for a portfolio's maximum return at the same level of portfolio risk as the benchmark while satisfying a given TE constraint.

Maximum Return (MR): reflects the portfolio with the highest expected return for a given TE (on the constant TE frontier) in mean-variance space. This portfolio sits on the apex of the ellipse in risk/return space. This high expected return, however, is accompanied by high levels of total portfolio risk.

Minimum Variance (MV): is situated at a point on the constant TE frontier with the minimum risk. It is independent of the benchmark, suffers from high levels of concentration (Chan, Karceski, & Lakonishok, 1999) and is exposed to considerable estimation error (Roncalli, 2014). Varadi, Kapler, Bee, and Rittenhouse (2012) found that the MV portfolio provides the least diversification efficiency, an unsurprising result given the excessive concentration of the least volatile assets with low correlations.

Jorion Benchmark Risk (JB): Jorion (2003) proposed that a constraint be imposed that total portfolio volatility be the same as that of the benchmark. This strategy takes advantage of the "flatness" of the constant TE frontier, resulting in portfolios with total risk equal to that of the benchmark and an expected return that is only marginally lower than those obtainable using the MR strategy.

Minimum Intra-Correlation (MIC): is a valuable measure of portfolio diversification (Livingston, 2013). Varadi et al. (2012) demonstrated the importance of minimizing average asset correlations to reduce portfolio variance (risk), thus, taking advantage of diversification through weighting lower correlating assets (diversifiers) to the rest of the portfolio. Varadi et

al. (2012) found not only a favorable risk-adjusted performance, but also a superior diversification efficiency (diversification indicator). The MIC portfolio is independent of the TE – this is the "Min IC no constraint" portfolio in Figure 2. The MIC portfolio, which is subject to a TE constraint, must be found by examining the results shown in Figure 3.

Maximum Diversification Ratio (MD): Choueifaty (2006) introduced the MD portfolio and the diversification ratio (DR). This strategy maximizes the degree of portfolio diversification and thereby results in portfolios, which have minimally correlated assets, lower risk levels and higher returns than other "traditional" portfolio strategies (Theron & van Vuuren, 2018). Portfolios with an MD ratio are maximally diversified and provide efficient alternatives to index tracking portfolios (Choueifaty, 2006). The maximum diversification ratio portfolio is independent of the TE – this is the "Max DR no constraint" portfolio in Figure 2. The MD ratio portfolio, which is subject to a TE constraint, must be found by examining the results shown in Figure 3.

Maximum Sharpe Ratio (MS): maximizing the Sharpe ratio of TE-constrained portfolios generates maximally risk-adjusted portfolios, entirely analogous to similar portfolios on the efficient frontier (Maxwell, Daly, Thomson, & van Vuuren, 2018).

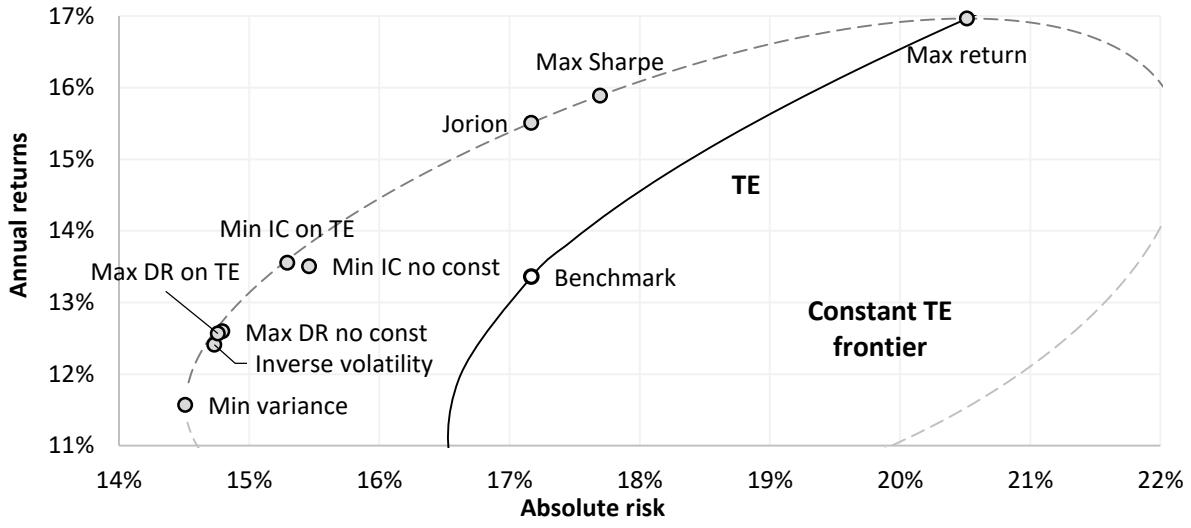
Historically, portfolio strategies have been evaluated on the efficient frontier. We introduce, for the first time, an evaluation of the performance of these six portfolio strategies by exploring their behavior in terms of performance ratios on the constant TE frontier.

2. DATA AND METHODOLOGY

2.1. Data

The data comprised simulated realistic weights, returns, volatilities and correlations for a small, standardized benchmark comprising three assets with the stylized parameters as provided in Table 1. This portfolio and associated benchmark obviously represent just one combination of an infinite series of possibilities. Portfolios and benchmarks with larger numbers of constituent assets were also explored and gave similar results. The important point is that this stylized combination of small portfolio and associated benchmark provides a numerical example for comparison and is like that used by Bajeux-Besnainou, Belhaj, Maillard, and Portait (2011).

Source: Jorion (2003) and own calculations.



Note: Here, *TE* = 7%, $r_f = 4\%$.

Figure 2. TE frontier, constant TE frontier and portfolio strategies

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Table 1. Stylized input data

Assets	A	B	C
Mean annual return	15%	19%	6%
Annual volatility	28%	25%	18%
Correlation matrix	1	0.09	0.16
	0.09	1	0.12
	0.16	0.12	1
Benchmark weights	50%	22%	28%

Portfolio constituents were derived only from the benchmark universe (and short-selling of benchmark constituents was permitted). This may constitute an unrealistic representation of what usually occurs in practice: most managed, TE-constrained portfolios are low-risk pension funds. The mandates governing investments in such funds will usually stipulate conservative investment strategies, so the high risks, which arise from short-selling, will not be permitted. See Daly and van Vuuren (2018) for a discussion on the differences between unconstrained and constrained (in the long/short sense) constant TE frontiers. Note that the "assets" which constitute the portfolio in the examples which follow could be asset classes (such as equity, bonds and cash), specific industry sectors within an asset class (e.g. an industrial equity index, a banking index, etc.) or individual assets such as single name stocks or bonds.

2.2. Methodology

To establish the methodologies required for the various frontiers, some definitions are first required. This subsection proceeds by introducing and describing the relevant variables and algebraic components. The mathematics governing the generation of the efficient frontier is then set out, followed by the algebra, which defines the TE frontier and then the constant TE frontier. Having built these foundations, the algebraic methodology, which is required to extract the portfolio weights for each possible strategy, is presented.

Active fund managers are tasked with outperforming specified benchmarks and the active asset positions they take may or may not be benchmark components (depending on the mandate governing the fund). The algebra required to derive the relevant investment strategy weights uses the same underlying variables, matrices and matrix notation defined below:

q: vector of benchmark weights for a sample of N assets;

x: vector of deviations from the benchmark;

$q_P\left(=q+x\right)$: vector of portfolio weights;

E: vector of expected returns;

σ: vector of benchmark component volatilities;

ρ: benchmark correlation matrix;

V: covariance matrix of asset returns;

r_f: risk-free rate.

Net short sales are allowed in this formulation, so the total active weight $q_i + x_i$ may be negative for any individual asset, i. The universe of assets can generally exceed the components of the benchmark, but for Roll's (1992) methodology, assets in the benchmark must be included. Expected returns and variances are expressed in matrix notation as:

$\mu_B = q'E$: expected benchmark return;

$\sigma_B^2 = q'Vq$: variance of benchmark return;

$\mu_\varepsilon = x'E$: expected excess return;

$\sigma_\varepsilon^2 = x'Vx$: TE variance (i.e. TE^2).

The active portfolio expected return and variance is given by:

$$\mu_P = \left(q+x\right)' E = \mu_B + \mu_\varepsilon, \tag{1}$$

$$\sigma_P^2 = \left(q+x\right)' V \left(q+x\right) = \sigma_B^2 + 2q'Vx + \\ + x'Vx = \sigma_B^2 + 2q'Vx + \sigma_\varepsilon^2. \tag{2}$$

The portfolio must be fully invested, defined as $\left(q+x\right)' 1 = 1$, where 1 represents an N-dimensional vector of $1s$. Using Merton's (1972) terminology, the following parameters are also defined: $a = E'V^{-1}E$, $b = E'V^{-1}1$, $c = 1'V^{-1}1$, $d = a - b^2/c$, $\Delta_1 = \mu_B - b/c$, where $b/c = \mu_{MV}$ and $\Delta_2 = \sigma_B^2 - 1/c$, where

$1/c = \sigma_{MV}^2$. Note that where the algebra allows for deviations from the benchmark to be calculated (x) these are presented, and the total portfolio component weights required are simply $q + x\left(= q_P\right)$. In other cases, where the investment strategy is unaffected by the imposition of a TE constraint, the relevant portfolio weights, w, are used instead.

Mean-variance frontier in absolute return space: minimize $q_P' V q_P$ subject to: $q_P' 1 = 1$ and $q_P' E = G$, where G is the target return. The vector of portfolio weights is determined using

$$q_P = \left(\frac{a - bG}{d}\right) q_{MV} + \left(\frac{bG - \frac{b^2}{c}}{d}\right) q_{TG}, \qquad (3)$$

where q_{MV} is the vector of asset weights for the MV portfolio given by $q_{MV} = V^{-1}1/c$ and q_{TG} is the vector of asset weights for the tangent (optimal) portfolio given by $q_{TG} = V^{-1}E/b$.

TE frontier: maximize $x'E$ subject to: $x'1 = 0$, $x'Vx = \sigma_\varepsilon^2$. The solution for the vector of deviations from the benchmark, x, is:

$$x = \pm\sqrt{\frac{\sigma_\varepsilon^2}{d}} V^{-1}\left(E - \frac{b}{c}1\right). \qquad (4)$$

Constant TE frontier: maximize $x'E$ subject to: $x'1 = 0$, $x'Vx = \sigma_\varepsilon^2$ and $\left(q + x\right)' V \left(q + x\right) = \sigma_P^2$. The solution for the vector of deviations from the benchmark, x, is:

$$x = -\frac{1}{\lambda_2 + \lambda_3} V^{-1}\left(E + \lambda_1 + \lambda_3 V q\right), \qquad (5)$$

where

$$\lambda_1 = -\frac{\lambda_3 + b}{c}, \qquad (6)$$

$$\lambda_2 = \pm(-2)\sqrt{\frac{d\Delta_2 - \Delta_1^2}{4\sigma_\varepsilon^2 \Delta_2 - y^2}} - \lambda_3, \qquad (7)$$

$$\lambda_3 = \left| = -\frac{\Delta_1}{\Delta_2} \pm \frac{y}{\Delta_2}\sqrt{\frac{d\Delta_2 - \Delta_1^2}{4\sigma_\varepsilon^2 \Delta_2 - y^2}}. \right. \qquad (8)$$

Jorion (2003) defined $z = \mu_P - \mu_B$ and $y = \sigma_P^2 - \sigma_B^2 - \sigma_\varepsilon^2$ and established that the relationship between y and z is $dy^2 + 4\Delta_2 z^2 - 4\Delta_1 yz - 4\sigma_\varepsilon^2\left(d\Delta_2 - \Delta_1^2\right) = \tilde{0}$, which describes an ellipse – a constant TE frontier – in return/risk space solving for z:

$$z\Big| = \frac{\Delta_1 y \pm \sqrt{\left(\Delta_1^2 - d\Delta_2\right)\cdot\left(y^2 - 4\Delta_2\sigma_\varepsilon^2\right)}}{2\Delta_2}. \qquad (9)$$

A highly risk-averse manager may opt for an MV portfolio, foregoing potential higher returns in exchange for the lowest possible risk, while a risky manager may select the MR portfolio at the expense of associated high risk. Managers seeking optimal risk/reward trade-offs would choose tangent (MS) portfolios (Maxwell et al., 2018), shown in Figure 1(b), while others may seek portfolios for which the diversification ratio is at a maximum, and still others might desire a MIC portfolio, or one which exhibited risk parity, etc. Note that component volatilities, correlations, expected returns and benchmark weights are all fixed, the variables in this exposition are the active portfolio weights.

Maximum Return (MR)

Jorion (2003) showed that the absolute maximum return on the constant TE frontier would be reached at the intersection of the TE frontier with the constant TE frontier, i.e. where $\mu_P = \mu_B + \sqrt{dT}$. The associated portfolio volatility is calculated using (2):

$$\sigma_P^2 = \sigma_B^2 + 2\Delta_1\sqrt{\frac{T}{d}} + \sigma_\varepsilon^2.$$

Note that this portfolio is equivalent to maximizing the IR given by $IR = \text{Excess return} / TE$. With a fixed TE, the maximum IR is reached when the numerator reaches a maximum, i.e. where $\mu_P = \mu_B + \sqrt{dT}$.

Jorion's Benchmark Risk (JB)

For this constraint, Jorion (2003) set $\sigma_P^2 = \sigma_B^2$ so that, from (2), this constraint implies that $2q'Vx = -\sigma_\varepsilon^2$. From (5), the vector of portfolio weight deviations from the benchmark, x, may be determined.

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Maximum Sharpe Ratio (risk-adjusted return) (MS)

Using the approach from Maxwell et al. (2018), the tangent portfolio (MS portfolio) on the constant TE frontier occurs where the slopes of the tangent line and the constant TE frontier are equal, i.e. where:

$$\frac{r_f-\mu_B}{\sigma_P^2}=\frac{\sqrt{\left(\Delta_1^2-d\Delta_2\right)\left[\left(\sigma_P^2-\sigma_B^2-\sigma_\varepsilon^2\right)^2-4\Delta_2\sigma_\varepsilon^2\right]}}{2\Delta_2\sigma_P^2}+$$
$$+\frac{\Delta_1\cdot\left(\sigma_P^2-\sigma_B^2-\sigma_\varepsilon^2\right)}{2\Delta_2\sigma_P^2}- \quad (10)$$
$$-\frac{\left(\Delta_1^2-d\Delta_2\right)\cdot\left(\sigma_P^2-\sigma_B^2-\sigma_\varepsilon^2\right)}{\sqrt{\left(\Delta_1^2-d\Delta_2\right)\left[\left(\sigma_P^2-\sigma_B^2-\sigma_\varepsilon^2\right)^2-4\Delta_2\sigma_\varepsilon^2\right]}}+\Delta_1$$
$$\frac{}{\Delta_2}.$$

Solving for μ_P and σ_P establishes the MS portfolio coordinates in return/risk space. Because these coordinates are unique, the weight deviations from the benchmark, x, are reverse engineered from (5).

Minimum Variance (MV)

Jorion (2003) showed that the active portfolio volatility is:

$$\sigma_P^2=\sigma_B^2\pm 2\sqrt{\frac{T}{d}}\left(\mu_B-\mu_{MV}\right)+\sigma_\varepsilon^2.$$

and that the absolute MV (of a portfolio subject to a TE constraint) is:

$$\sigma_P^2=\sigma_B^2-2\sqrt{T\left(\sigma_B^2-\sigma_{MV}^2\right)}+\sigma_\varepsilon^2.$$

The associated expected return is calculated using (1).

Maximum Diversification (MD)

The diversification ratio was introduced by Choueifaty (2006) and is defined as:

$$MD=\frac{\left(q+x\right)'\sigma}{\sqrt{\left(q+x\right)'V\left(q+x\right)}}. \quad (11)$$

MD portfolios have a vector of active portfolio weights (Pemberton & Rau, 2007):

$$x_{MD}=\frac{V^{-1}\sigma}{\sigma' V^{-1}\sigma}-q.$$

These weights generate a universal, non-TE-constrained, MD portfolio (so it is not necessarily on, or inside, the constant TE frontier). A closed-form solution for a TE-constrained MD portfolio has not been identified, but such a portfolio may be identified empirically. Using (11), active portfolio weights, x, which define the efficient TE-constrained set were used to calculate the DR at various σ_P values. The active portfolio weights, which generate the maximum DR, are easily identified.

Risk parity/inverse volatility

Portfolios in which the risk contribution from each component is made equal is a form of diversification maximization, because such portfolios are like minimum variance portfolios subject to diversification constraints on component weights (Maillard, Roncalli, & Teiletche, 2010). The components weights are:

$$w_i=\frac{1}{n\beta_i},$$

where β_i are the component β_S and n is the number of assets of which the portfolio consist. The problem of endogeneity arises here, since is a function of the component β_i which in turn depends on the portfolio composition (i.e. w_i). Various iterative numerical solutions are used (Maillard, Roncalli, & Teiletche, 2010).

The TE constraint does not affect these weights – the constituents of the weights are affected by the number of constituents and their respective β. Neither of these are altered by imposing a TE constraint. These portfolios are included for comparison only. Inverse volatility portfolios are similar in construction. The portfolio weights are assembled in proportion to the inverse of their volatility, so

$$w_i=\frac{\frac{1}{\sigma_i}}{\sum_i^n\frac{1}{\sigma_i}}, \quad (12)$$

where σ_i are the individual component volatilities. This approach ignores the correlation between asset components and again the TE constraint does not affect these (shown here for comparison only).

Source: Own calculations.

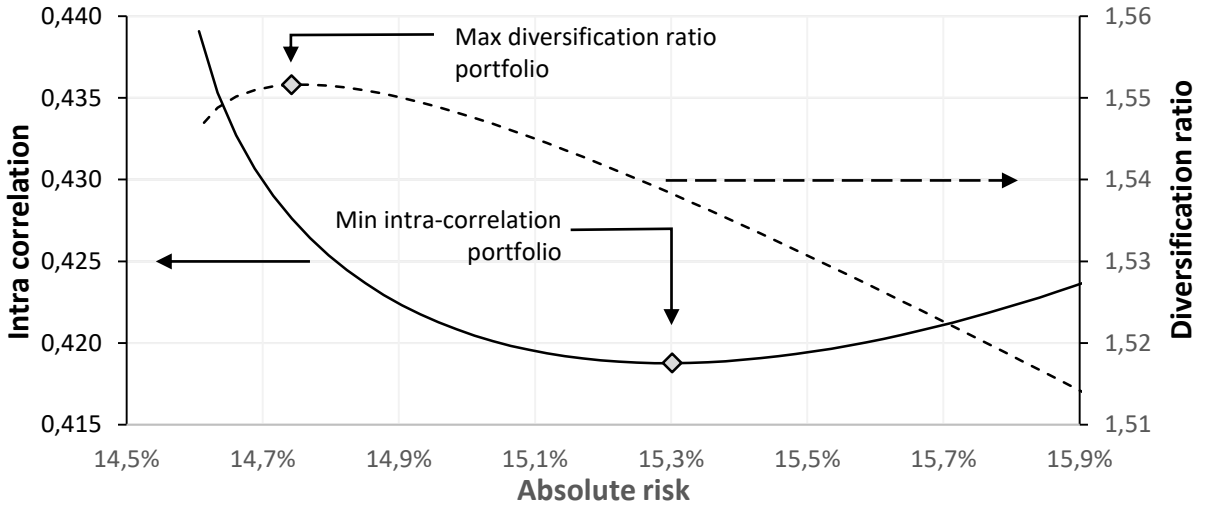


Figure 3. Intra-correlation and diversification ratio subject to a *TE* = ٪٧

Minimum Intra-Portfolio Correlation (MIC)

There are competing definitions of intra-portfolio correlation, but the one used here avoids most of the problems associated with the measure (Livingstone, 2013):

$$MIC\Big| = \frac{\Sigma_i \Sigma_j (q+x)_i (q+x)_j \rho_{ij}}{\Sigma_i \Sigma_j (q+x)_i (q+x)_j} = \tag{13}$$

$$= \left[(q+x)'(q+x)\right]^{-1}(q+x)' \times$$

$$\times \left[\rho - \text{diag}(\rho)\right](q+x),$$

where $\text{diag}(\rho)$ is the matrix of the diagonal elements of ρ.

As with the MD portfolio, the MIC as defined above is a universal, non-TE-constrained, MIC portfolio (so, again, it is not necessarily on or inside the constant TE frontier). A closed-form solution for a TE-constrained MIC portfolio must be found empirically. Active portfolio weights, x, which define the efficient TE-constrained set are used to calculate the MIC at different values for σ_P. The active portfolio weights, which generate the minimum MIC, are easily identified. For a

Source: Jorion (2003) and own calculations.

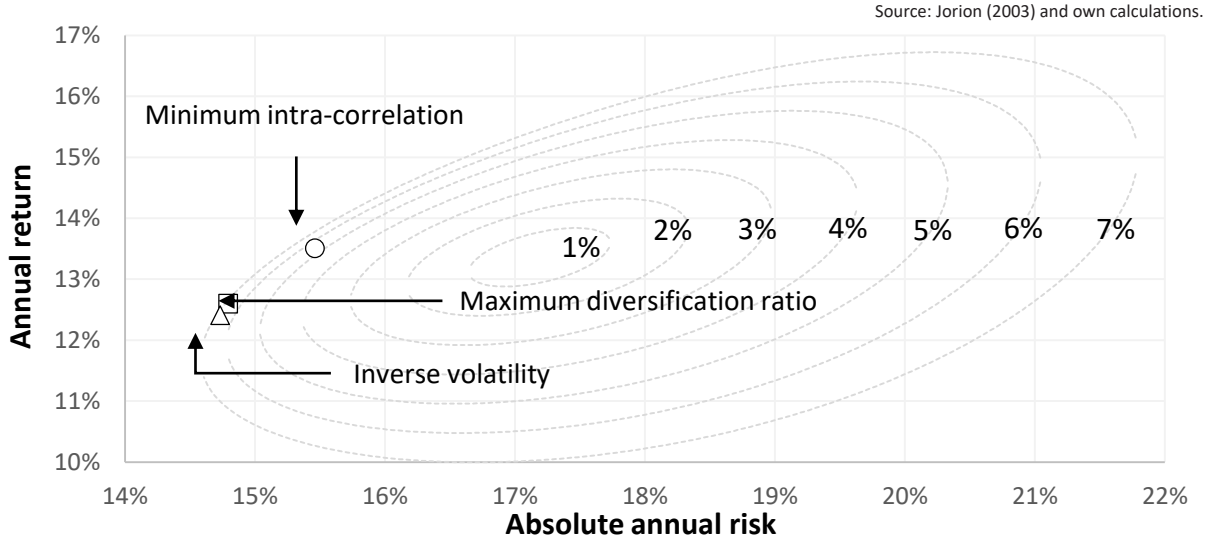


Figure 4. The empirical identification of applying brute force mathematics on the MD and MIC portfolios to be constrained by TE

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constant $TE = 7\%$, the intra-correlation and MD portfolio returns/standard deviations are shown in Figure 3.

Figure 4 shows the position for this stylized example of the overall MIC, the inverse volatility and the MD portfolios. These coordinates in risk-returns space are not influenced by the TE.

3. RESULTS AND DISCUSSION

Figure 5 illustrates the movement of the constant TE frontier about the benchmark for incremental increases of TE in annual risk/return space.

Figure 5 describes the loci of the six portfolio strategies as a function of 1% increases in TE. Increasing the TE expands the ellipse (constant TE frontier) outward in a balloon-like fashion, with the benchmark (initially) remaining at the centre of the ellipse. Once the TE increases above 6%, however, the ellipse reaches the boundary of the global efficient frontier (utility maximizing investment sets). Here, the ellipse nestles into the curve of the efficient frontier, expanding right with each incremental increase in TE. At higher TEs, the benchmark shifts away from the ellipse's centre. This is a simple observation that can easily elude portfolio managers, as increasing TEs above 18% will result in the benchmark falling outside the ellipse. In a scenario where such high TEs are permitted, portfolio managers would invest in financial instruments that are significantly different from that of the benchmark.

Figure 5 illustrates how the MS and MR portfolio strategies are monotonically increasing for every increase in TE. The MR portfolio travels a more constant gradient as greater risk is compensated

Source: Own calculations.

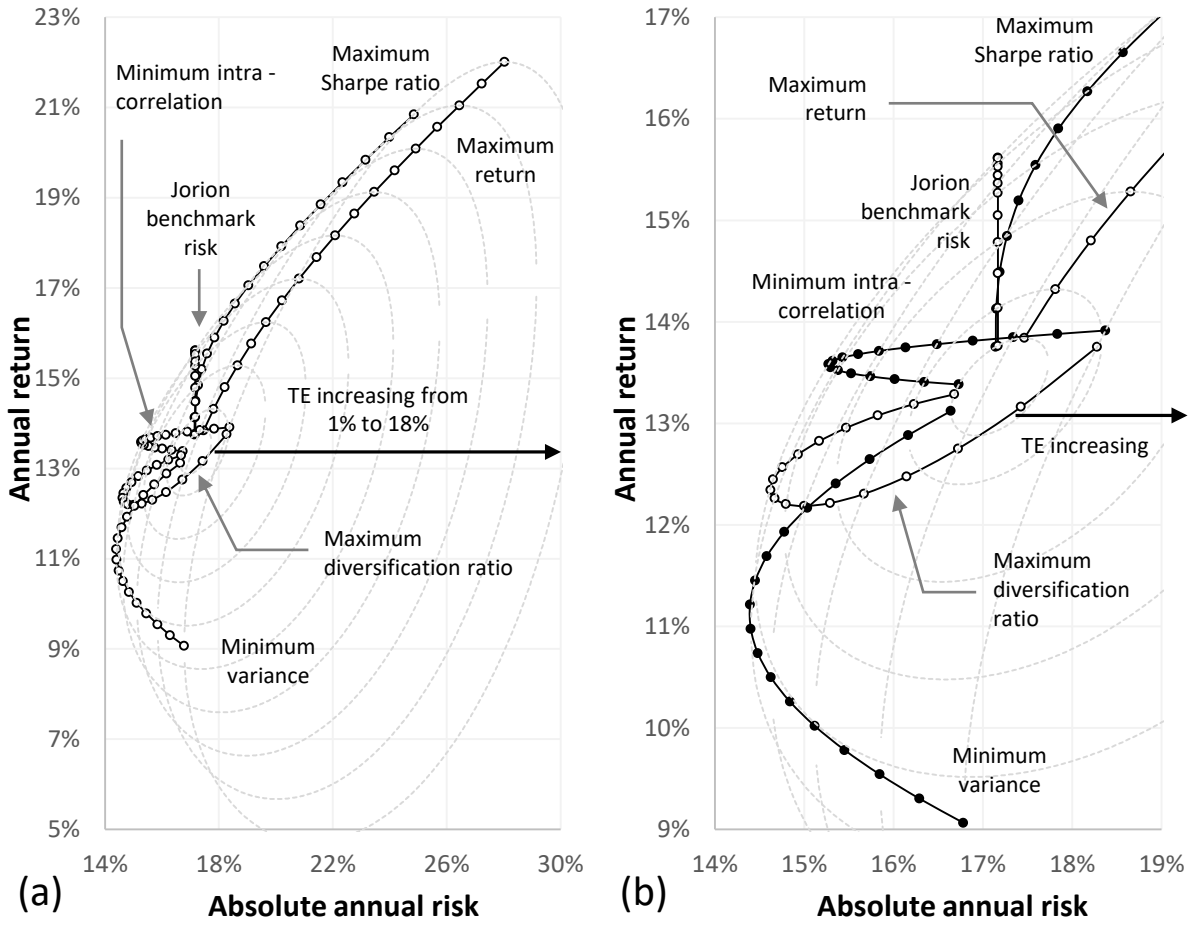


Figure 5. (a) Loci of the six portfolio strategies with respect to changes in their risk/return relationships for increasing TEs and (b) an enlarged section of Figure 4(a)

with steady increases in return. The MV portfolio is located on the leftmost edge of the constant TE frontier, for increasing TEs. Initially, as TE increases, the annual risk of the MV portfolio strategy decreases, but this risk reduction is short lived as higher TEs result in a more rapid increase in risk, forming a left shaped parabola, where increased TEs diminish investor utility. The MV portfolio is tangential to the global MV portfolio at this turning point, thus, reaching a ceiling for maximizing risk-averse investor utility.

The principle flaw of the MV portfolio strategy, however, is due to the continuous decrease in annual returns through all TEs. Thus, portfolio managers following an MV portfolio strategy will experience initial risk-reducing effects with the opportunity cost of expected returns foregone, however, by selecting higher TEs, the overall risk/return relationship decreases. The turning point of the MV portfolio strategy is due to the combined impact of the nestling effect of the constant TE frontier and the eastward shift away from the global MV portfolio discussed above.

The JB portfolio strategy maintains a fixed risk level (per definition). As the constant TE frontier expands with increasing TE, the annual return of this portfolio strategy increases, but at a TE of approximately 9%, the JB portfolio strategy reaches an apex and then decreases. Fixing the level of risk to the benchmark decreases the portfolio's ability to produces dynamic returns as higher TE leads to the benchmark falling outside the constant TE frontier.

As TE increases, the MIC portfolio displays a slow increase in annual returns and rapid risk reduction. For higher values of TE, portfolio managers will experience a plateau in annual returns, accompanied by rapidly increasing annual risk. Lowering intra-portfolio correlations of the MIC portfolio will provide risk-reducing benefits, however, to achieve greater annual returns the intra-portfolio correlations need to increase.

Although the MD reflects a similar risk/return position to that of MV portfolio for low TEs, as the TE increases, the path of the MD portfolio decreases in parallel with the MV portfolio's southwest bound direction, displaying risk-reducing characteristics at the expense of decreasing annual returns. Like the MV portfolio, the MD portfolio experiences a change in direction where risk increases. Unlike, the MV portfolio, the MD portfolio exhibits a rapid increase in annual returns, surpassing not only the initial MD annual returns of lower TEs but tending beyond the annual returns of the MIC portfolio strategy. The reason for this turning point may be in the relationship between excess return generation and TE.

Lower TEs result in decreasing portfolio returns, which erodes the risk-return trade-off of the MD portfolio. However, the more rapidly decreasing risk leads to better performance as the level of TE increases, thus, portfolio returns increase more rapidly with higher incremental increases in TE. Higher TEs allow the MD portfolio to benefit from the tilt and shift of the constant TE frontier. The IR may provide better reasoning for this turning point.

The previous work of Choueifaty (2006) on the MD portfolio strategy and the combined efforts of Hedge Fund Consistency Index (2011) and Livingstone (2013) on the MIC portfolio strategy defined these strategies as universal, non-TE-constrained portfolios (i.e. they are not necessarily on or inside the constant TE frontier). Due to this independence from the constant TE frontier, a closed-form solution for TE-constrained MD and TE-constrained MIC portfolios has never been identified, but such a portfolio may be identified empirically. This article introduces a method to manipulate the equations of the MD and MIC portfolios onto the constant TE frontier.

Figure 6(a) illustrates the performance of the six TE constrained portfolio strategies in terms of their respective Sharpe ratios, for given TEs, thus, identifying the point at which each portfolio strategy achieves its unique MS (i.e. the level of TE for which each portfolio strategy performs best). Figure 6(b) illustrates the performance of the six ratios in terms of their M^2 ratio.

The MV portfolio exhibits the poorest performance at the lowest TEs. This poor performance only tends to decrease with higher TEs, resulting in the worst performing portfolio. This is most likely due to the low return achieved by the port-

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Source: Sharpe (1966) and own calculations.

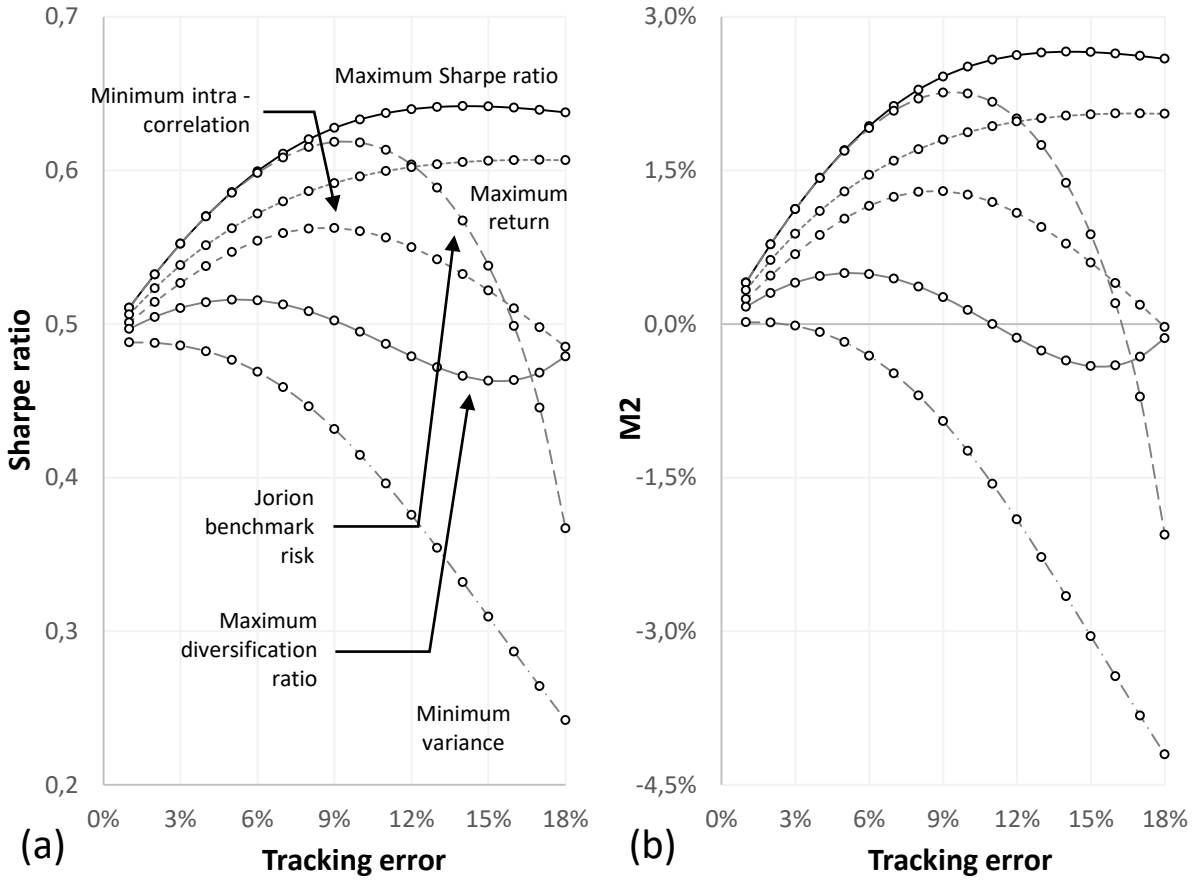


Figure 6. Performance evaluation of the six portfolio strategies with respect to their (a) Sharpe ratios and (b) M^2 ratios for incremental increases in TE

folio in relation to the risk-free rate (i.e. as total portfolio risk increases, the low returns result in performance being eroded). The MD portfolio displays an unusual result as performance increases steadily for $1\% \geq TE \geq 5\%$. Beyond this point, the MD experiences a temporary performance decrease until a TE of $\approx 15\%$, where the portfolio experiences an unanticipated increase in its Sharpe ratio. This increase in performance may be due to the greater level of diversification (i.e. lower correlation with the benchmark), leading to greater portfolio returns.

The MIC portfolio ranks third in terms of performance at lower TEs, reaching its MS at a TE of approximately 8%. However, performance decreases for higher TEs. This may suggest that the level of intra-portfolio correlation reaches a limit, where the level of correlation can only be effective until a specified point. This would explain why higher TEs risk may lead to decreased performance.

Although MR displays the expected result of consistent positive performance, the Sharpe ratio reaches a plateau of approximately 0.61. This may be the result of the precise moment where the benchmark falls outside the constant TE frontier, as such high TEs reflect portfolio positions that are far off from that of the benchmark. Another unusual performance was found in the JB portfolio strategy. Initially, the TE interval of 1-6% showed signs of competition between JB and MS for the first-place performance ranking until JB's Sharpe ratio was maximized at a TE of 9%. TEs beyond this point resulted in JB suffering the most performance decline among the portfolios tested, dropping to the fifth-place rank.

The fixed risk constraint for the JB portfolio increases portfolio performance, however, without the adoption of increased risk, returns cannot continue to increase, leading to the reversal in performance. The most consistent and best-performing portfolio was the MS ratio portfolio, which

displayed similar characteristics to the MR portfolio (i.e. increasing monotonically until reaching higher TEs, where performance plateaus). This impressive performance is likely due to the balanced trade-off between risk and return, as maximizing the Sharpe ratio (for increasing TEs) provides a more dynamic portfolio.

The more comprehensive and interpretable Modigliani risk-adjusted performance measure (L. Modigliani & F. Modigliani, 1997), the M^2 ratio, measures risk-adjusted returns relative to a benchmark:

$$M^2 = \overline{r}_f + \left(\frac{\overline{R}_p - \overline{r}_f}{\sigma_p} \right) \sigma_M . \qquad (14)$$

Although this measure provides a similar performance ranking to the Sharpe ratio, L. Modigliani and F. Modigliani (1997) explained that the M^2 measure has the advantage of being more interpretable as it provides a percentage for outperformance. A positive M^2 ratio signifies benchmark outperformance.

In a comparison between the Sharpe (Figure 6(a)) and M^2 (Figure 6(b)) measurements, Figure 7(a) tells a more comprehensive story about the benchmark outperformance of these portfolio strategies. The MV portfolio underperforms the benchmark for all levels of TE, rendering the strategy redundant to an active portfolio manager. JB and MIC provided benchmark outperformance for lower levels of TE, however, were not as resilient for higher TEs. Figure 7(a) illustrates how the MD strategy provides low benchmark outperformance for low TEs. MR and MS provided clear benchmark outperformance for all TEs. Figure 6(b) provides great insight to investors in the evaluation of active managers.

Figure 7(a) evaluates the performance of the six TE-constrained portfolios to their respective IRs. Figure 7(b) shows the correlation of each portfolio strategy with the benchmark. Active managers must determine how much risk relative to the benchmark should be installed to achieve the desired outperformance. TE indicates the level of this benchmark risk, however, we have presented an alternative measure of benchmark risk in the analysis of the relationship between *TE* and the correlation between the benchmark and portfolio return (Ammann & Tobler, 2000):

$$TE = \sqrt{\sigma_P^2 \cdot \left(1 - \rho^2\right)} \text{ so } \rho = \sqrt{1 - \left(\frac{TE}{\sigma_P} \right)^2} .$$

Source: Own calculations.

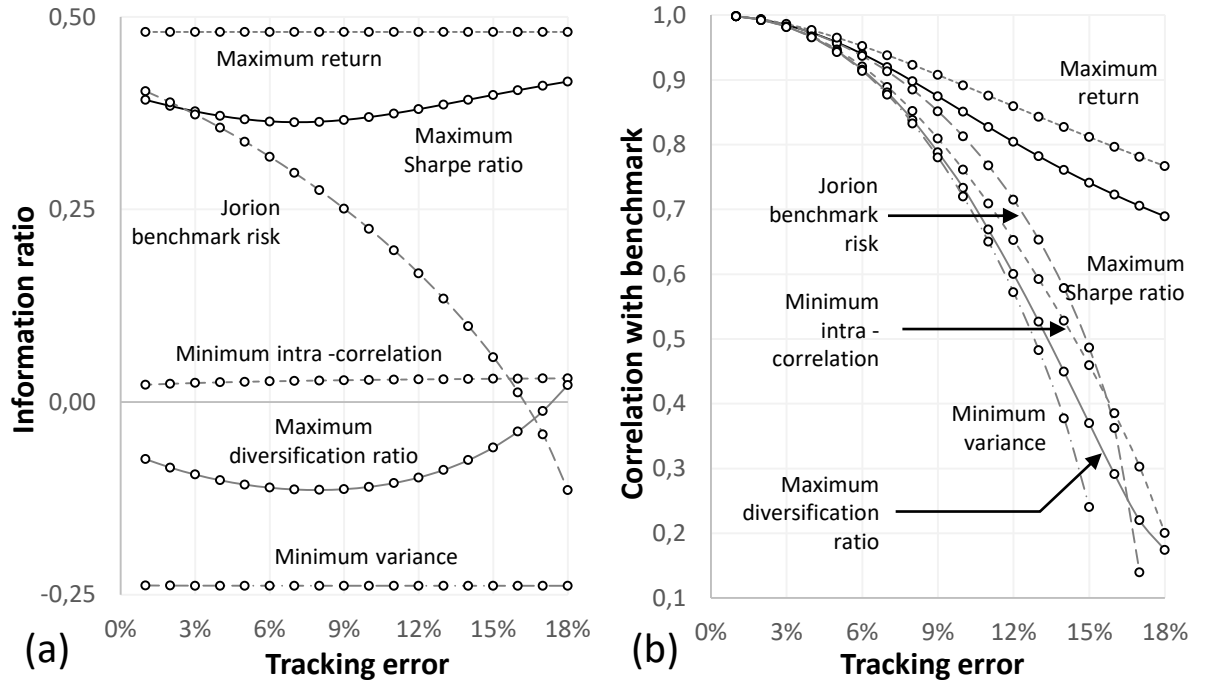


Figure 7. (a) Performance of the six portfolio strategies with respect to their IRs and (b) correlation of benchmark and each strategy's returns, as a function of *TE*

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Note that the correlation depends on the tracking error variance, TE^2 and the portfolio variance, σ_P^2. $TE = 0$ implies $\rho = 1$. The correlation with the benchmark tends towards zero for higher levels of TE due to the decreasing level of benchmark exposure (i.e. as β decreases from 1 to 0, ρ between the portfolios and the benchmark weakens. This adds to Ammann and Tobler's (2000) analysis: as *TE* increases ρ falls and the rate of decrease is greater for smaller β. Also, as TE increases, benchmark risk approaches 0. This explains the superior performance of the MS and MR portfolios as *TE* increases – since they embrace total risk. Although a correlation coefficient of zero correctly indicates high risk, i.e., no benchmark exposure at all $\beta = 0$, $\rho = 1$ does not necessarily imply $\sigma_B = 0$. Instead, $\rho = 1$ might indicate $TE = 0$ or infinite benchmark exposure $\beta \to \infty$. Therefore, ρ is an inappropriate benchmark risk measure unless the portfolio is unleveraged, i.e., unless $\beta < 1$.

The *IR* is the quotient of the expected active return and the *TE*, therefore, indicates how much active return a portfolio manager can expect at a specified level of tracking risk. *IR* reveals the manager's ability and consistency to generate excess returns and is represented by:

$$IR = \frac{r_p - r_B}{TE}. \tag{15}$$

The MR and MV portfolios displayed unsurprising performance in Figure 7, resulting in the best and worst performance, respectively. The MR portfolio provided increasing returns that consistently dominated the returns of the risk-free asset, revealing how incrementally increasing the TE (as the denominator of this equation), can indeed lead to increased performance. Figure 7 shows some valuable information on the performance comparison between the JB and MS portfolios, as TEs of below 2% favor the JB's portfolio. The dominant performance of the JB portfolio at low TEs is likely due to risk being fixed to the benchmark, which in turn is lower than that of the MS portfolio. The MS portfolio experiences declining performance at lower TEs, likely due to tracking the inefficient benchmark positions too closely, however, an upswing occurs at TEs above 8%, narrowing the gap between MS and MR.

Although JB's performance beats that of the MS portfolio at the lowest TEs, Figure 8 confirms the inevitable value reduction of increasing TE on the JB portfolio. MIC experiences flat levels of performance, indicating unresponsiveness to TE adjustments. Finally, Figure 8 confirms the unexpected performance upswing of the MD portfolio for $TE \geq 8\%$. This may confirm the inefficiency of the benchmark itself, leading to undiversified positions. Therefore, as TE increases away from the benchmark, optimal levels of diversification are

Source: Own calculations.

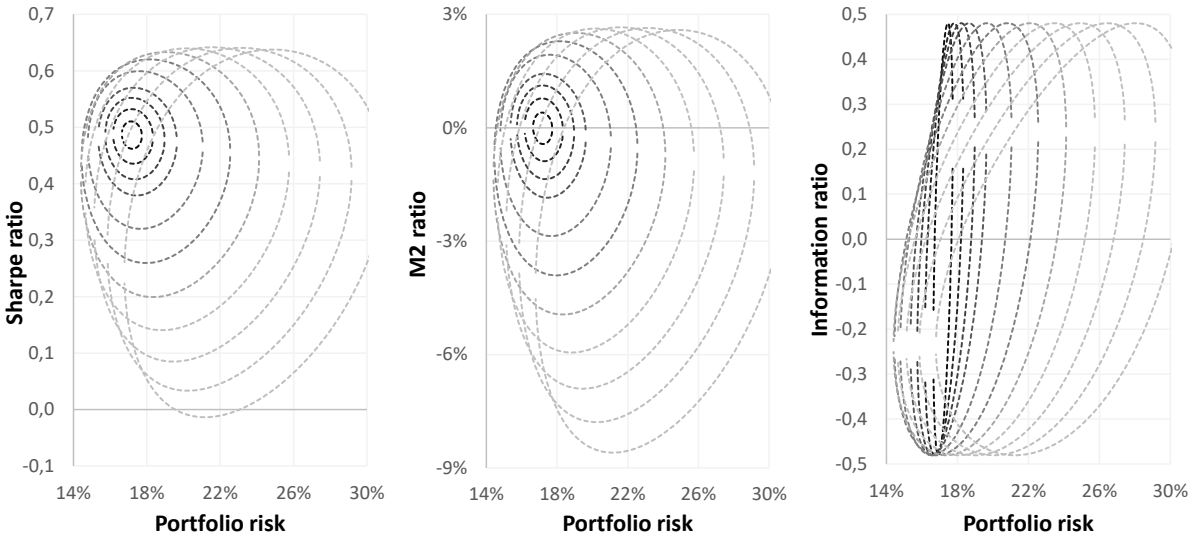


Figure 8. The loci of possible (a) Sharpe ratios, (b) M^2 ratios and (c) IRs – as a function of portfolio risk – for increasing *TE*s

Source: Own calculations.

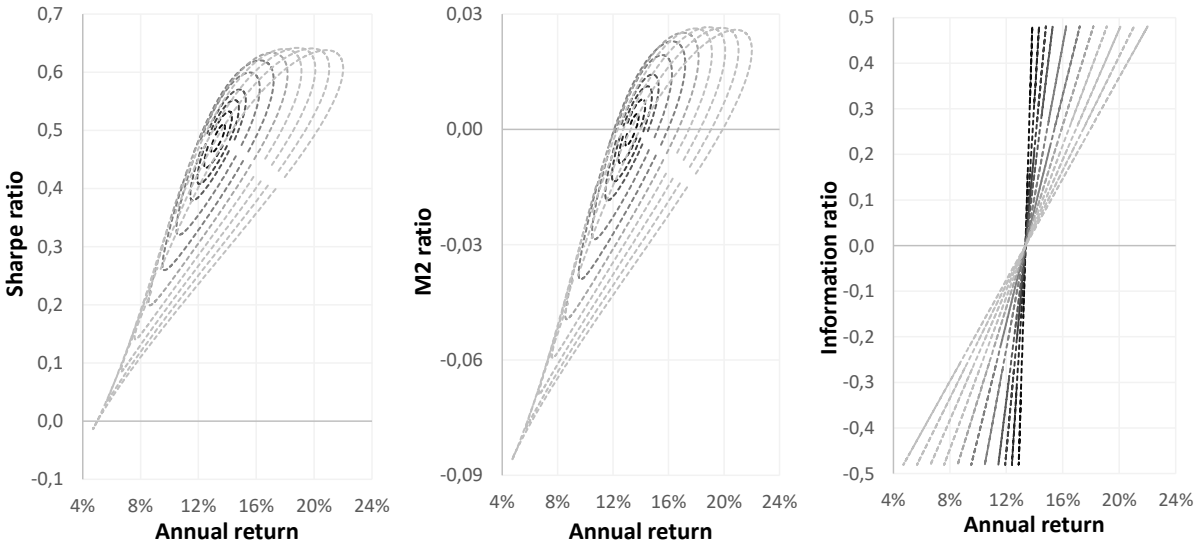


Figure 9. The loci of possible (a) Sharpe ratios, (b) M^2ratios and (c) IRs – as a function of portfolio return – for increasing *TE*s

achieved resulting in an improved performance of the MD portfolio.

Figures 8 and 9 show the loci of performance ratios (Sharpe, M^2 and Information) – as a function of portfolio risk and as a function of portfolio returns – for increasing TEs, respectively.

Increasing TE between $1\% \le TE \le 5\%$ (i.e. $0.51 \le \text{Sharpe ratio} \le 0.58$) inflates the constant TE ellipse around the benchmark in absolute risk/return space. Increasing TE between $6\% \le TE \le 10\%$ (i.e. $0.60 \le \text{Sharpe ratio} \le 0.64$) pushes the west quadrant of the constant TE ellipse into the efficient frontier such that it becomes flush with it. The

Source: Own calculations.

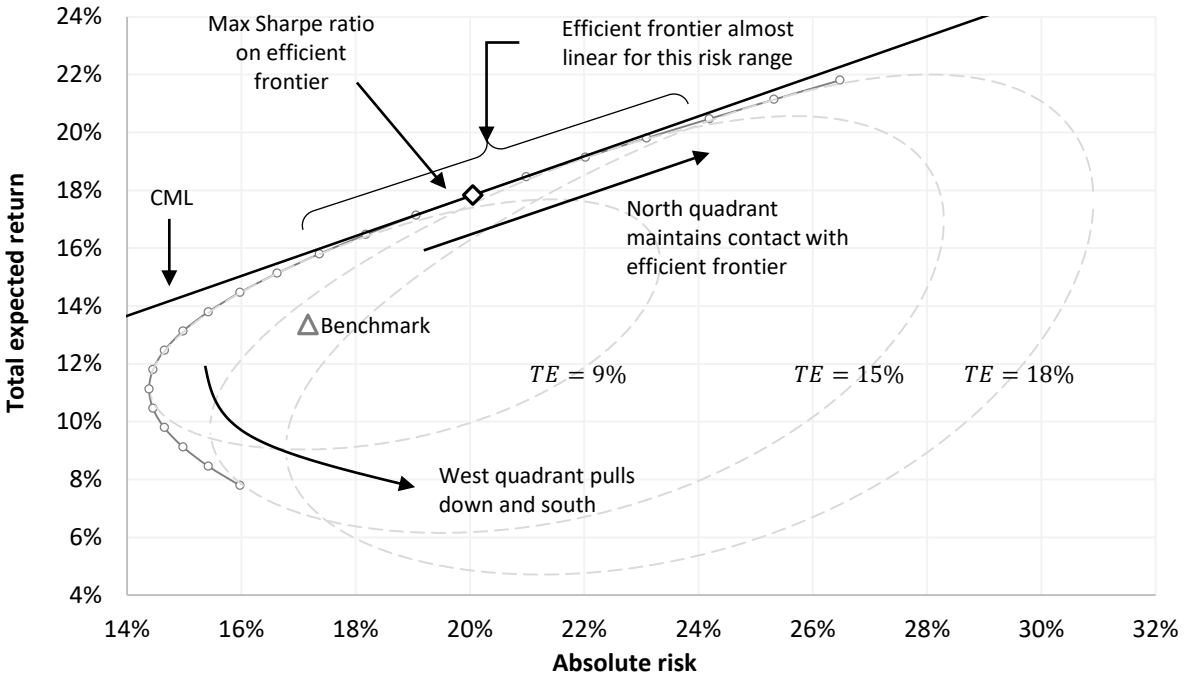


Figure 10. Explanation of performance ratio plateaus as $TE \to 9\% \to 15\% \to 18\%$. The efficient frontier is the darker grey line to which the CML is tangent

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Sharpe ratio reaches a plateau (0.64) for $TE \geq 10\%$. $TE \geq 10\%$. The reason for this is as follows: for high TEs (i.e. $TE \geq 10\%$), the constant *TE* ellipse retreats from the efficient frontier such that the west quadrant shifts rightwards. The north quadrant also retreats rightwards, but remains in contact with the efficient frontier at roughly the position of the MS portfolio. Higher TEs, then, do not generate higher Sharpe ratios: the maximum slope of the CML, by definition, cannot increase (Figure 10).

This explanation also holds for the M^2 ratio, as well as the IR ceiling. The maximum IR obtainable without a TE constraint (i.e. on the efficient frontier) is the same as the maximum IR with a TE constraint for large TEs. If TEs are sufficiently large, the constant TE ellipse is tangential to the efficient frontier at high returns and associated high risks (Figure 10).

CONCLUSION AND SUGGESTIONS FOR FUTURE WORK

The inefficiency of capital markets provides investment opportunities for active management strategies. Recent (2018) evidence has identified a shift by well-informed investors towards this investment philosophy. Greater emphasis on benchmark outperformance is required for active management. Investors use benchmarks as a relative performance measure to determine the value added by portfolio managers relative to the risks undertaken. The components and weights of these benchmarks, however, are arbitrarily chosen, leading to a gauge of performance that is frequently inefficient and leads, ultimately, to sub-optimal portfolio selection.

Active management brings with it active risk. TE, as a measure for active risk, is not a metric used in isolation, but rather in combination with other performance measures. Although the inherent flaws of TE constraints are documented, the investment management industry maintains adherence. The introduction of the constant TE frontier allowed active managers to explore the effect of imposing additional constraints on active strategies and thus to mitigate these effects. Because not all investors share identical risk preferences or objectives, defining an "optimal" strategy becomes paramount. Various investment strategies have been suggested to satisfy investors' risk appetites.

Historically, performance measures have been used to evaluate strategies on the efficient frontier – a pursuit which makes sense when investment in the universe of investable securities is permitted. This article used several performance indicators to evaluate benchmark outperformance of six active portfolio strategies (MR, MV, JB, MD, MIC and MS) subject to a TE constraint – on the constant TE frontier.

The performance of the six TE constrained portfolio strategies varied considerably. The MS portfolio achieved the greatest level of performance, as higher excess returns result in lower absolute risk relative to the benchmark. JB exhibits competitive features at lower TEs, but suffers a substantial decline for higher TEs. The MIC and MV portfolios display similar performances, moving in a parallel fashion for lower TEs in risk/return space. The MD portfolio exhibited unusual performance: performing poorly at lower TEs, but showing an improved risk/return relationship for higher TEs, which lead to benchmark outperformance.

The performance ratios reach plateaus for high TEs because of the roughly linear nature of the efficient frontier in this region of risk/return space. Because the constant TE ellipse remains in contact with the efficient frontier for high TEs, the MS for the former will always be approximately the same as the latter. The efficient frontier Sharpe ratio is, of course, a universal maximum: no better risk-adjusted return portfolio exists. Similar arguments hold for the other performance indicators. In this work, the stylised example did not consider long-only investment strategies. Real world funds face short-selling restrictions because of the riskiness of this approach. Future work could tighten up this requirement.

Other restrictions on investment policies are common in contracts between investors and fund managers. These restrictions might stipulate that the share of certain types of assets should be smaller, higher or equal to a given percentage. These constraints are often inherent to the fund's investment strategy as specified in the fund's prospectus. Examples include an industry sector fund might specify that chief investments are in its corresponding sector; funds dedicated to prudent investors (such as pensions) may specify upper bounds on equity holdings or lower limits on governmental bond holdings, and so on. Weights constraints can also be set by regulators and funds with tax benefits or tax-deferred funds are frequently subject to weight restrictions. None of these were considered in this analysis, but results gathered from strategies, which considered these restrictions, could provide fruitful to fund managers and investors alike.

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