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Continuous time evolutionary market dynamics: the case of fix-mix strategies

Abstract

We develop a continuous-time evolutionary market model where prices are endogenously generated by supply and demand. Investment strategies are assumed to be fix-mix, which means that the relative budget shares are constant in time. The model is therefore a hybrid. While given portfolio rules remain constant over time, assets, market-clearing and in particular market shares of the individual portfolio strategies evolve in continuous time. Our main goal is to understand the wealth dynamics which describes the evolution of market shares. We study its asymptotic properties and identify evolutionary stable investment strategies. These strategies prevent entrants to the financial market from gaining wealth in the long run and furthermore, in the existence of a small diversified number of mutant strategies, drive the invading strategy out of the market. Our definition of evolutionary stability is therefore a close adaptation of Maynard-Smith and Price's (1973) original definition of an ESS [8].

Keywords: evolutionary economics, evolutionary finance, continuous-time portfolio theory, endogenously determined asset prices, evolutionary stability of trading strategies.
JEL Classification: G11, D52, D81, E2.

Introduction

The Darwinian principle "Survival of the Fittest" is classically used by evolutionary biologists to explain the origination and evolution of species. The two key principles in Darwin's theory are mutation and selection. It is now more and more recognized that the dynamic of many economic systems relies on these exact principles. Full rationality, which is missing in the biological context, has mostly disappeared as a key assumption in economic models. Mutation can be explained by the existence of noise traders or traders which rationally decide to try out a new strategy, which they believe is superior to the market strategy. The process of selection can be understood as a process of adaptation and imitation, rather than a process of inheritance in evolutionary biology. On the other hand, the bankruptcy of a company using a particular market strategy very much corresponds to the case of extinction of an inferior species and it can be argued that among all economic systems, financial markets are probably the ones which closest resemble biological systems.

Evolutionary financial market models in discrete time have been considered in Blume and Easley [2], Evstigneev, Hens and Schenk-Hoppe [3], [4], [6], and Farmerand Lo [5]. The main point in setting up an evolutionary financial market model is the specification of an evolutionary dynamic which determines the market shares of the relevant trading strategies over time. In general such a dynamic will depend on the stochastic payoffs, dividends or prices of the underlying assets, as well as the trading strategies, which are assumed to be adapted to the underlying information structure. The models developed by Evstigneev, Hens and Schenk-Hoppe assume that stochastic dividends resp. payoffs of the underlying assets are exogenously given, but that in contrast to other models the asset prices are determined by the trading strategies and a market clearing condition. In this article we use a similar approach as in Evstigneev, Hens and Schenk-Hoppe but set up a model in continuous time rather than in discrete time. The choice of continuous time brings with it the usual technical problems which lie in the analytical formulation of the model, in particular in a probabilistic framework, but has the major benefit, that methods from classical analysis such as partial differential equations and stochastic calculus become applicable and provide powerful tools for the problems solution. We therefore think that it is necessary to adapt discrete time evolutionary finance models to a continuous time framework. In this article we present a first step into this direction by considering a model in which the trading strategies are assumed to be of such a type, that the relative budget shares are constant in time.The same

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type of trading strategies has been used in Evstigneev, Hens and Schenk-Hoppe [4]. The model can then be considered as a hybrid. While given portfolio rules remain constant over time, assets, market-clearing and in particular market share of the individual portfolio strategies evolve in continuous time. The case of more general strategies will be considered in future work. Nevertheless, we set up a continuous time stochastic dynamic for this model which describes the evolution of market shares in a population of finitely many trading strategies. Mathematically our dynamic presents a random differential equation. A mathematical treatment of such objects can be found in Arnold [1] or Soong [9], to which we also refer at times. Under this dynamic and the assumption of ergodic dividend processes we show that the portfolio rule which invests according to expected relative dividends is evolutionary stable and hence that a market consisting entirely of traders using this rule can not be invaded by a small number of traders using different rules. Additionally we derive the optimal strategy for invasion into a market which is not yet dominated by the evolutionary stable strategy and hence not yet evolutionary mature. The structure of the article is as follows. In section 2 we set up a discrete time market selection dynamic with variable time discretization Δt, while in section 3 we determine the corresponding continuous time dynamic by considering the limit of this dynamic for $\Delta t \to 0$. In section 4 we derive the evolutionary stable strategy and the optimal invasion strategy. The main conclusions of the article are summarized in section 5.

1. The discrete model

In this section we adapt the discrete time evolutionary finance model developed by Evstigneev, Hens and Schenk-Hoppe [3] in a way that we introduce time steps of arbitrary length Δt as opposed to time steps of fixed length equal to one. Furthermore we interpret payoffs of assets as consumptions goods rather than dividends or real payoffs. This has the nice benefit that our model and conclusions are also valid in a general macroeconomic context, where assets simply need to be replaced by production units. The re-formulation of the model allows us to study the limit behavior of the resulting dynamic for $\Delta t \to 0$. More precisely, we consider an economy consisting of $k = 1,...,K$ assets which all produce the same consumption good. The price of this consumption good at time t is denoted by p_t^0. The amount of the consumption good produced by asset k in the period $[t, t+\Delta t)$ is denoted by $D_{t+\Delta t}^k$. A potential investment strategy is given by $(\lambda_{t,0},...,\lambda_{t,K})_t$, where $\lambda_{t,0}$ represents the consumption at time t in units of the consumption good, and $\lambda_{t,k}$ is the fraction of the wealth, which the investor assigns to the purchase of the k-th asset in period $[t, t+\Delta t)$. We note that $\lambda_{t,k}$ is determined at the beginning of the period $[t, t+\Delta t)$, while $D_{t+\Delta t}^k$ is determined at the end. This justifies the choice of sub-indices. We assume that our economy is influenced by $i = 1,...,I$ investment strategies $(\lambda_t^i)_t$ but that all these strategies use the same consumption rate $c > 0$, i.e.

$$\lambda_{t,0}^i = c\Delta t \tag{1}$$

for all i. In general the trading strategies may depend in a nontrivial way on time and the state of the economy. In this article we assume, however, that the investment strategies are constant in time. More precisely we assume that

$$\lambda_{t,k}^i = \lambda_k^i(1 - c\Delta t) \tag{2}$$

for all t with constants λ_k^i satisfying $\sum_{k=1}^K \lambda_k^i = 1$ and $0 < \lambda_k^i < 1$ for $i = 1,...,I$. Our investment strategies are therefore completely diversified. The factor $1 - c\Delta t$ which represents the consumption has been introduced in order to guarantee that $\sum_{k=0}^K \lambda_{t,k}^i = 1$. Note that the expression in equation (2) depends on the size Δt of the time step and we therefore use the notation

$$\lambda_k^i(\Delta t) := \lambda_k^i(1 - c\Delta t) \tag{3}$$

Finally we denote with $w_{t+\Delta t}^i$ the wealth generated by the i-th investment strategy and with

$$w_{t+\Delta t} = \left(w_{t+\Delta t}^1, \cdots, w_{t+\Delta t}^I\right)$$

the wealth vector at time $t+\Delta t$. Market clearing prices $p_{t+\Delta t}^k$ for the assets $k = 1,...,K$ are determined from the following condition:

$$p_{t+\Delta t}^k = \lambda_k(\Delta t) w_{t+\Delta t} \tag{4}$$

Let us then denote with

$$\theta_{t,k}^i = \frac{\lambda_k^i(\Delta t) w_t^i}{p_t^k} = \frac{\lambda_k^i(\Delta t) w_t^i}{\lambda_k(\Delta t) w_t} = \frac{\lambda_k^i w_t^i}{\lambda_k w_t} \tag{5}$$

the number of units of asset k held by portfolio rule i during the period $[t, t+\Delta t)$. Then we obtain the following wealth equation:

$$w_{t+\Delta t}^i = \sum_{k=1}^K \left(p_{t+\Delta t}^0 D_{t+\Delta t}^k + \lambda_k(\Delta t) w_{t+\Delta t}\right)\theta_{t,k}^i \tag{6}$$

which is analogous to equation (2.7) in Evstigneev, Hens and Schenk-Hoppé [3]. Now it follows directly from the market clearing condition on consumption goods that

$$\sum_{i=1}^{I}\sum_{k=1}^{K} p^0_{t+\Delta t} D^k_{t+\Delta t}\theta^i_{t,k} = \sum_{i=1}^{I} c\Delta t w^i_{t+\Delta t}. \tag{7}$$

Noting that

$$\sum_{i=1}^{I}\theta^i_{t,k} = 1 \tag{8}$$

we obtain from (7) that

$$p^0_{t+\Delta t} D_{t+\Delta t} = c\Delta t W_{t+\Delta t}, \tag{9}$$

where

$$D_{t+\Delta t} \equiv \sum_{k=1}^{K} D^k_{t+\Delta t}, W_{t+\Delta t} \equiv \sum_{i=1}^{I} w^i_{t+\Delta t}. \tag{10}$$

As in Hens and Schenk-Hoppé [6] we assume that $D_t \geq 0$ for all times t and all states ω. In order to compare different strategies, it is more convenient to consider market shares of portfolio rules instead of the amount of the actual wealth. Denoting by

$$r^i_t = \frac{w^i_t}{W_t}$$

the market share of portfolio rule i, we conclude easily from (5) and (9) that

$$r^i_{t+\Delta t} = \sum_{k=1}^{K}\left(c\Delta t d^k_{t+\Delta t} + \lambda_k(\Delta t) r_{t+\Delta t}\right)\theta^i_{t,k}, \tag{11}$$

where

$$d^k_{t+\Delta t} \equiv \frac{D^k_{t+\Delta t}}{D_{t+\Delta t}} \tag{12}$$

is the relative dividend payment of asset k and $r_{t+\Delta t} \equiv (r^1_{t+\Delta t}, \cdots, r^I_{t+\Delta t})'$. Let us introduce the following notation

$$\Theta_t = \left(\theta^i_{t,k}\right)_{I\times K}, \Lambda = \left(\lambda^i_k\right)_{K\times I}.$$

Then we infer from (11) that

$$r_{t+\Delta t} = c\Delta t\left[Id - (1-c\Delta t)\Theta_t\Lambda\right]^{-1}\left(\Theta_t d_{t+\Delta t}\right) \tag{13}$$

where $d_{t+\Delta t} \equiv (d^1_{t+\Delta t}, \cdots, d^K_{t+\Delta t})'$ and Id represents the $I \times I$ identity matrix.

Remark 1. *Under our assumptions a minor modification of the proof of Proposition 1 in [3] shows that the market share dynamic (13) is well-defined and in particular the matrix* $Id - (1-c\Delta t)\Theta_t\Lambda$ *is invertible for every* $\Delta t > 0$.

2. A continuous time evolutionary market model

In this section we establish a continuous-time evolutionary market model by considering the limit of (13) for $\Delta t \to 0$. We will use the following notation:

$$A = \left(\alpha_{i,j}\right)_{I\times I} \equiv \Theta_t\Lambda.$$

It then follows from Lemma A1 in the technical appendix that the market share dynamics (13) can be expressed as

$$r^i_{t+\Delta t} = \frac{\begin{vmatrix} 1+(c\Delta t-1)a_{1,1} & \cdots & z^1_{\Delta t} & \cdots & (c\Delta t-1)a_{1,I} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ (c\Delta t-1)a_{I,1} & \cdots & z^I_{\Delta t} & \cdots & 1+(c\Delta t-1)a_{I,I} \end{vmatrix}}{\begin{vmatrix} 1 & \cdots & 1 & \cdots & 1 \\ (c\Delta t-1)a_{2,1} & \cdots & (c\Delta t-1)a_{2,i} & \cdots & (c\Delta t-1)a_{2,I} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ (c\Delta t-1)a_{I,1} & \cdots & (c\Delta t-1)a_{I,i} & \cdots & 1+(c\Delta t-1)a_{I,I} \end{vmatrix}} \tag{14}$$

where $z_{\Delta t} = (z^i_{\Delta t})_{I\times 1} \equiv \Theta_t d_{t+\Delta t}$. On account of

$$\sum_{i=1}^{I} z^i_{\Delta t} = 1,$$

it follows from equation (14) and equation (23) in the technical appendix that

$$\lim_{\Delta t\to 0^+} r^i_{t+\Delta t} = \frac{B_{1,i}}{|B|} \tag{15}$$

where

$$B \equiv \left(b_{i,j}\right)_{I\times I} = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ -a_{2,1} & 1-a_{2,2} & \cdots & -a_{2,I} \\ \vdots & \vdots & \cdots & \vdots \\ -a_{I,1} & -a_{I,2} & \cdots & 1-a_{I,I} \end{pmatrix},$$

and $B_{1,i}$ is the algebraic cofactor of the element $b_{1,i}$ in the matrix B. The following Proposition is useful under two aspects. First, it establishes that our construction of the wealth dynamic has a continuous time limit, and, second, it establishes an alternative formula for the market share vector r_t.

Proposition 1. *We have the following relationship for the limit of the market shares dynamics with respect to* $\Delta t \to 0$

$$\lim_{\Delta t\to 0^+} r^i_{t+\Delta t} = \frac{B_{1,i}}{|B|} = r^i_t, i = 1, \cdots, I. \tag{16}$$

From the point of classical evolutionary game theory, in particular with hindsight of the definition of an evolutionary stable strategy (ESS) in the sense of Maynard Smith and Price (1973) we can often restrict our analysis to the case of only two strategies, i.e. $I = 2$. The situation is then interpreted in a way that the market is dominated by one particular strategy, potentially the ESS strategy, and due to be invaded (or not) by another strategy. Such a restriction does not allow us to explain any features which relate to co-evolution, in which two strategies influence each other positively and are jointly driving out a third one, which is individually superior than the first two. We admit that a discussion of co-evolutionary market dynamics is necessary at some point, but for now keep up with the existing line of classical evolutionary game theory models.

By means of Proposition 1 we are now able to determine the random differential equation which describes the evolution of market shares in continuous time. As indicated, we focus on the particular case $I = 2$.

Proposition 2. *Assume that the relative dividend processes* $(d_t^k)_{t>0}, k = 1,\cdots,K$ *are exogenously given processes with continuous paths, then the stochastic market shares* r_t^1 *and* r_t^2 *corresponding to two investment strategies satisfy the following random differential equations*

$$\frac{dr_t^1}{dt} = \frac{cz_0^1 - cr_t^1}{a_{1,2} + a_{2,1}} = c\frac{\left(\sum_{k=1}^K \frac{d_t^k \lambda_k^1}{\lambda_k^1 r_t^1 + \lambda_k^2 (1-r_t^1)}\right) - 1}{\sum_{k=1}^K \frac{\lambda_k^1 \lambda_k^2}{\lambda_k^1 r_t^1 + \lambda_k^2 (1-r_t^1)}} r_t^1 \qquad (17)$$

$$\frac{dr_t^2}{dt} = -\frac{dr_t^1}{dt} = \frac{cz_0^2 - cr_t^2}{a_{1,2} + a_{2,1}} = c\frac{\left(\sum_{k=1}^K \frac{d_t^k \lambda_k^2}{\lambda_k^1 (1-r_t^2) + \lambda_k^2 r_t^2}\right) - 1}{\sum_{k=1}^K \frac{\lambda_k^1 \lambda_k^2}{\lambda_k^1 (1-r_t^2) + \lambda_k^2 r_t^2}} r_t^2. \qquad (18)$$

Both equalities hold a.s.

Equations (17) and (18) describe the selection process inherent in our market model. The mutation feature will be implicitly assumed in our definition of evolutionary stability which will follow in the next section. Equations (17) and (18) are well defined in the sense that solutions for arbitrary initial conditions exist and are unique in the mean square sense.

Proposition 3. *There exists a unique mean square solutions in the sense of Arnold and Song [1], [9] for both random differential equations (17) and (18) and arbitrary initial conditions* $r_0^1, r_0^2 \in [0,1]$.

The following Corollary which follows from the uniqueness part or Proposition 3 guarantees that there are no sudden deaths or bankruptcies in our economy.

Corollary 1. *The hyperplane is invariant under the dynamic (17) and (18). This means that given* $r_0^i = 0$, *i* = 1,2, *then* $r_t^i = 0$ *for all* $t \geq 0$; *and given* $r_0^i > 0, i = 1,2$, *then* $r_t^i > 0$ *(a.e.) for all* $t \geq 0$. *Therefore there are no sudden deaths or bankruptcies in our economy.*

3. Evolutionary stability of the market

We assume in this section that the relative dividend processes $(d_t^k)_{t>0}, k = 1,\cdots,K$ are first-order stationary and ergodic, i.e. $E(d_t^K) =: \overline{d}_k$ is time-independent for all $k = 1,\cdots,K$ and

$$\overline{d}_k = \lim_{T\to\infty} \frac{1}{T}\int_0^T d_t^K dt$$

holds with probability 1. Under this assumption we show that a portfolio rule which invests according to the relative expected dividends payed by the assets is evolutionary stable and hence can not be invaded by other fix-mix strategies. For discrete time evolutionary market models similar results have been obtained in Hens and Schenk-Hoppé [3]. The assumption on first order stationarity is satisfied for example if the relative dividend process follows a martingale, which economically represents a reasonable assumption. The second condition above corresponds to the fact that ergodicity of stochastic processes intuitively describes the property that no sample helps to meaningfully predict values that are very far away in time from that sample and that the time path of the stochastic process is not sensitive to initial values. For technical reasons we further assume that the expectations of the relative dividend processes $\overline{d}_k, k = 1,\cdots,K$ are all strictly positive.

As noted in the proof of Corollary 1, there exist two fixed points (1,0) and (0,1) in (17) and (18) respectively. By matters of symmetry we can without loss of generality restrict our discussion on stability below to the stability of the fixed point (1,0) of (17) and (18), which corresponds to the solution $r_t^1 = 1, r_t^2 = 0, t \geq 0$. Obviously we have $r_t^1 = 1 - r_t^2$ and therefore it is sufficient to study equation (18) exclusively. We think of the portfolio rule λ^1 as the incumbent market strategy which is due to be invaded by the portfolio rule λ^2.

Definition 1. *The exponential growth rate* $g(\lambda^2;\lambda^1)$ *of the portfolio rule* λ^2 *'s wealth share in a market dominated by the portfolio rule* λ^1 *is defined via*

$$g(\lambda^2;\lambda^1) \equiv \lim_{T\to\infty} \frac{1}{T} \log\left(\frac{r_T^2}{r_0^2}\right). \quad (19)$$

The existence of the limit in Definition 1 strongly depends on the ergodicity assumption on the relative dividends. We omit the technical details. Using the exponential growth rate as a substitute for real payoffs in classical game theory and applying a dynamical picture rather than a static one, the direct implementation of Maynard Smith and Price's idea of an evolutionary stable strategy [8] leads us to the following definition:

Definition 2. *A portfolio rule* λ^{ESS} *is called evolutionary stable, if the following conditions hold:*

1) *for any fix-mix portfolio rule* $\lambda \neq \lambda^{ESS}$ *we have* $g(\lambda;\lambda^{ESS}) \leq 0$ *;*
2) *if* $\lambda \neq \lambda^{ESS}$ *and* $g(\lambda;\lambda^{ESS}) = 0$ *then there exists a fix-mix portfolio rule* λ^{mut} *s.t.* $g(\lambda^{mut};\lambda) > 0 \geq g(\lambda^{mut};\lambda^{ESS})$.

The first condition says that under the market selection process, a potential invader can not grow if the market is in evolutionary equilibrium. The second condition says, that in the presence of a mutation process, the potential invader is in fact driven out of the market by the ESS strategy. The connection between various static versions of ESS and their stability properties, when applying some sort of evolutionary dynamic has been widely studied. See for example Weibull [10] for a good overview of the classical theory. As our market shares do not correspond to payoff in the classical sense and our definition of ESS is not standard, these results do not apply directly to our case. It was shown in [6] in a discrete time framework that the growth rate determines the local stability of the fixed point $(1,0)$ in (17), (18). This proof can be adapted to cover the case of continuous time random dynamical systems by making use of the stochastic Hartman-Grobmann Theorem (Arnold [1], p. 377). We omit the details. We find the following theorem:

Theorem 1. *The portfolio rule* λ^* *defined by* $\lambda_k^* \equiv \overline{d}_k$ *with* $\overline{d}_k = \mathrm{E}(d_t^k)$ *is an evolutionary stable portfolio rule.*

Having determined the evolutionary stable market rule, we now turn to a slightly different problem. In a market which has not yet reached evolutionary equilibrium, what is the best rule to invade the market. Such a result is in fact highly interesting when the aim is to set up a different market dynamic in which agents switch strategies according to their success in the sense of an adaptive dynamics (see Hofbauer, [7], chapter 9) rather than strategies competing for capital which is the line taken in this article. We find the following theorem:

Theorem 2. *Provided that the portfolio rule* λ^2 *dominates the market, the best portfolio rule to invade the market* λ^{inv} *is given by*

$$\lambda_k^{inv} = \frac{\sqrt{\lambda_k^2 \overline{d}_k}}{\sum_{i=1}^{K} \sqrt{\lambda_i^2 \overline{d}_i}}, k = 1,\cdots,K. \quad (20)$$

The exponential growth rate of λ^2 *'s wealth share is* $g(\lambda^2;\lambda^{inv}) \leq 0$ *and* $g(\lambda^2;\lambda^{inv}) = 0$ *if and only if* $\lambda^2 = \lambda^*$.

Conclusions

In this article we derive a continuous time version of the evolutionary market models introduced in Evstigneev, Hens and Schenk-Hoppé [3] and [6]. We think that a continuous time approach, though technically more demanding and difficult to handle, bears in prospect the benefits of analytical techniques such as partial differential equations and stochastic calculus. Our model presents a first step into this new direction and is elementary in a way that we concentrate ourselves on so called fix-mix strategies. Nevertheless the model is accurate and provides interesting insights. The evolutionary stable investment strategy among all fix-mix strategies as well as an optimal market invasion strategy are derived. The restriction on fix-mix strategies will be relaxed in future research work. Nevertheless, though simple in its structure our model is powerful enough as to explain the well known market strategy "invest according to expected dividends" from an evolutionary point of view and therefore our model indicates the right way to go.

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Appendix A

Lemma A1. *The market shares dynamics (13) is given by*

$$r_{t+\Delta t} = \frac{\left(Id-(1-c\Delta t)A\right)^{*}\Theta_t d_{t+\Delta t}}{\begin{vmatrix} 1 & 1 & \cdots & 1 \\ (c\Delta t-1)a_{2,1} & 1+(c\Delta t-1)a_{2,2} & \cdots & (c\Delta t-1)a_{2,I} \\ \vdots & \vdots & \cdots & \vdots \\ (c\Delta t-1)a_{I,1} & (c\Delta t-1)a_{I,2} & \cdots & 1+(c\Delta t-1)a_{I,I} \end{vmatrix}} \tag{21}$$

where "$*$" represents the matrix adjoint operator.

Proof. We conclude from (13) and Cramer's rule that

$$r_{t+\Delta t} = c\Delta t\frac{\left[Id-(1-c\Delta t)\Theta_t\Lambda\right]^{*}}{\left\|\left[Id-(1-c\Delta t)\Theta_t\Lambda\right]\right\|}\left(\Theta_t d_{t+\Delta t}\right) \tag{22}$$

and furthermore from (5) that

$$\sum_{i=1}^{I}\theta_{t,k}^{i} = 1, \sum_{i=1}^{I} a_{i,j} = 1, j = 1,\cdots,I. \tag{23}$$

Hence the sum of each column in the matrix $\left[Id-(1-c\Delta t)\Theta_t\Lambda\right]$ is equal to $c\Delta t$. Finally, the lemma can be directly derived from the properties of the determinant function.

Proof ***(Proposition 1).*** We assume w.l.o.g that $I = 2$. The general case can be proved similarly, but is notational much more demanding. It follows directly from (17) that

$$r_{t+\Delta t}^{1} = \frac{\begin{vmatrix} z_{\Delta t}^{1} & (c\Delta t-1)a_{1,2} \\ z_{\Delta t}^{2} & 1+(c\Delta t-1)a_{2,2} \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ (c\Delta t-1)a_{2,1} & 1+(c\Delta t-1)a_{2,2} \end{vmatrix}} \tag{24}$$

and thus from $z_{\Delta t}^{1} + z_{\Delta t}^{2} = 1$ that

$$r_{t+\Delta t}^{1} = \frac{c\Delta t z_{\Delta t}^{1} - (c\Delta t-1)a_{1,2}}{(a_{1,2}+a_{2,1}) + c\Delta t(a_{2,2}-a_{2,1})}. \qquad r_{t+\Delta t}^{2} = 1 - r_{t+\Delta t}^{1} \tag{25}$$

From the last equation we obtain

$$\lim_{\Delta t\to 0^{+}} r_{t+\Delta t}^{1} = \frac{a_{1,2}}{a_{1,2}+a_{2,1}}.(a.s)$$

Noting that

$$a_{1,2} = \left(\sum_{k=1}^{K}\frac{\lambda_k^1\lambda_k^2}{\lambda_k r_t}\right)r_t^1, \tag{26}$$

and

$$a_{2,1} = \left(\sum_{k=1}^{K} \frac{\lambda_k^1 \lambda_k^2}{\lambda_k r_t} \right)(1 - r_t^1), \tag{27}$$

we get

$$a_{1,2} + a_{2,1} = \sum_{k=1}^{K} \frac{\lambda_k^1 \lambda_k^2}{\lambda_k r_t} = \frac{a_{1,2}}{r_t^1}, \tag{28}$$

from which the statement of the proposition follows.

Proof ***(Proposition 2).*** It follows from equation (25) and de L' Hopital's theorem that

$$\lim_{\Delta t \to 0} \frac{r_{t+\Delta t}^1 - r_t^1}{\Delta t} = \frac{cz_0^1 - cr_t^1}{a_{1,2} + a_{2,1}} .(a.s.)$$

Thus (17) follows from both (28) and the definition of $z_{\Delta t}$. Equation (18) can be verified analogously.

Proof ***(Proposition 3).*** Clearly it is sufficient to prove there exists a unique mean square solution to (18). To reach this goal, define the function

$$f(X) \equiv c \frac{\left(\sum_{k=1}^{K} \frac{d_t^k \lambda_k^2}{\lambda_k^1 (1-X) + \lambda_k^2 X} \right) - 1}{\sum_{k=1}^{K} \frac{\lambda_k^1 \lambda_k^2}{\lambda_k^1 (1-X) + \lambda_k^2 X}} X \tag{29}$$

then the random differential equation (18) can be written as

$$\frac{dX}{dt} = f(X), X_0 = r_0^2, \in [0,1]. \tag{30}$$

Noting that the function $f(\cdot)$ is continuously differentiable in $[0,1]$ and $f(0) = f(1) = 0$, we infer that the phase space of (30) is a subset of interval $[0,1]$ for every $\omega \in \Omega$. For this reason, thanks to Theorem 5.1.2 in Soong (1973) [9], we only need to show that the function $f : \Xi \to L_2$ satisfies the following mean square Lipschitz condition:

$$f(X) - f(X) \| \leq \alpha \| X - Y \|, \alpha > 0, \tag{31}$$

where Ξ represents the set of all random variables with values in [0,1] and L_2 the corresponding space with norm $\|X\| = (\mathbf{E}(X^2))^{1/2}$. In order to prove (31), we note that since $X \in [0,1]$ we have

$$0 < \min(\lambda_k^1, \lambda_k^2) \leq \frac{\lambda_k^1 \lambda_k^2}{\lambda_k^1 (1-X) + \lambda_k^2 X} \leq \max(\lambda_k^1, \lambda_k^2), k = 1, \cdots, K. \tag{32}$$

For every $X, Y \in \Xi$, we have

$$|f(X) - f(Y)| \leq \left| cX \left(\frac{\left(\sum_{k=1}^{K} \frac{d_t^k \lambda_k^2}{\lambda_k^1 (1-X) + \lambda_k^2 X} \right) - 1}{\sum_{k=1}^{K} \frac{\lambda_k^1 \lambda_k^2}{\lambda_k^1 (1-X) + \lambda_k^2 X}} - \frac{\left(\sum_{k=1}^{K} \frac{d_t^k \lambda_k^2}{\lambda_k^1 (1-Y) + \lambda_k^2 Y} \right) - 1}{\sum_{k=1}^{K} \frac{\lambda_k^1 \lambda_k^2}{\lambda_k^1 (1-Y) + \lambda_k^2 Y}} \right) \right| + \left| c \frac{\left(\sum_{k=1}^{K} \frac{d_t^k \lambda_k^2}{\lambda_k^1 (1-Y) + \lambda_k^2 Y} \right) - 1}{\sum_{k=1}^{K} \frac{\lambda_k^1 \lambda_k^2}{\lambda_k^1 (1-Y) + \lambda_k^2 Y}} (X - Y) \right|. \tag{33}$$

Define

$$\lambda_m \equiv \min\{\lambda_k^1, \lambda_k^2 \mid k = 1, \cdots, K\}, \lambda_M \equiv \max\{\lambda_k^1, \lambda_k^2 \mid k = 1, \cdots, K\}, \tag{34}$$

where $0 < \lambda_m \leq \lambda_M < 1$. Accordingly, after lengthy but elementary calculations, it follows from (32) and (33) that

$$|f(X) - f(Y)| \leq \frac{2c\lambda_m \lambda_M + c\lambda_M^2}{K\lambda_m^4} |X - Y|. \tag{35}$$

Thus (31) is satisfied with

$$\alpha = \frac{2c\lambda_m \lambda_M + c\lambda_M^2}{K\lambda_m^4}, \tag{36}$$

Proof ***(Theorem 1).*** The linearization of (18) at $(1,0)$ is given by

$$\frac{dr_t^2}{dt} = c\left(-1+\sum_{k=1}^{K}\frac{\lambda_k^2}{\lambda_k^1}d_t^k\right)r_t^2. \tag{37}$$

Therefore, in a small neighborhood of (1,0), it follows from (37) and our assumption on ergodicity of the relative dividend processes that the exponential growth rate is given by

$$g(\lambda^2;\lambda^1) = c\left(-1+\sum_{k=1}^{K}\frac{\lambda_k^2}{\lambda_k^1}\overline{d}_k\right). \tag{38}$$

If $\lambda_k^1 = \overline{d}_k, k = 1,\cdots,K$, then $g(\lambda^2;\lambda^1) = 0$ no matter how the portfolio rule λ^2 is chosen. Nevertheless we see that condition 1 in Definition 2 is satisfied for λ^*. Now if $\lambda \neq \overline{d} = \lambda^*$, then there must exist a $k_0 \in \{1,\cdots,K\}$ such that

$$\frac{\overline{d}_{k_0}}{\lambda_{k_0}} - 1 > 0.$$

Consequently we can choose a fix-mix portfolio rule λ^{mut} as follows: Let $\lambda_{k_0}^{mut}$ be sufficiently close to 1 and all $\lambda_k^{mut}, k \neq k_0$ be very small. In this way we obtain a fix-mix portfolio rule which satisfies

$$g(\lambda^{mut};\lambda) = c\left(-1+\sum_{k=1}^{K}\frac{\lambda_k^2}{\lambda_k}\overline{d}_k\right) > 0.$$

and hence condition 2 of Definition 2 is also satisfied.

Proof ***(Theorem 2).*** Maximization of the growth rate for the invading rule is equivalent to minimizing the growth rate of the incumbent rule. Essentially we then have to solve the following nonlinear programming problem:

$$\min_{x_k;k=1,\cdots,K} -1+\sum_{k=1}^{K}\frac{\lambda_k^2}{x_k}\overline{d}_k \tag{39}$$

$$s.t.\begin{cases}\sum_{k=1}^{K} x_k = 1, \\ 0 < x_k < 1, k = 1,\cdots,K.\end{cases} \tag{40}$$

Since the target function is concave and the constraint convex, there exists a unique global optimal solution to (39). Defining the Lagrangian function

$$L(x_1,\cdots,x_K;\mu) \equiv -1+\sum_{k=1}^{K}\frac{\lambda_k^2}{x_k}\overline{d}_k + \mu\left(\sum_{k=1}^{K}x_k - 1\right),$$

and setting

$$\frac{\partial L(x_1,\cdots,x_K;\mu)}{\partial x_k} = 0, k = 1,\cdots,K. \tag{41}$$

It follows that

$$x_k = \sqrt{\frac{\lambda_k^2\overline{d}_k}{\mu}}. \tag{42}$$

Consequently we conclude from $\sum_{k=1}^{K} x_k = 1$ that the unique optimal solution to (39) is given by

$$x_k^* = \frac{\sqrt{\lambda_k^2\overline{d}_k}}{\sum_{i=1}^{K}\sqrt{\lambda_i^2\overline{d}_i}}, k = 1,\cdots,K.$$

By virtue of the optimality of x^*, we infer that

$$g(\lambda^2; x^*) \leq g(\lambda^2; \overline{d}) = 0. \quad (43)$$

Furthermore we show that $g(\lambda^2; x^*) < 0$ if $\lambda^2 \neq \overline{d}$. In order to reach this goal, we only need to prove $x^* \neq \overline{d}$ (thanks to the uniqueness). In fact, if on the contrary, $x^* = \overline{d}$ we obtain from (20) and

$$\sum_{k=1}^{K} \lambda_k^2 = \sum_{k=1}^{K} x_k = 1 \quad (44)$$

that

$$\mu = 1, \lambda_k^2 = \overline{d}_k, k = 1, \cdots, K.$$

However this is a contradiction to the assumption of $\lambda^2 \neq \overline{d}$. Thus $g(\lambda^2; x^*) < 0$.