“Advanced operational risk modelling in banks and insurance companies”

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Advanced operational risk modelling in banks and insurance companies

Abstract

The aim of this paper is to measure operational risk in financial institutions when historical data are available starting from a fixed threshold.

To quantify the operational risk we apply the Loss Distribution Approach (LDA), a frequency/severity approach widely used in the actuarial models. Risk measures like Value at Risk (VaR) and Expected Shortfall (ES) are used for determining the risk capital necessary to cover the operational risk.

The dependence among the events in the operational risk management has been taken into account using copula functions. We employed for this purpose the Student copula, which is widely used in financial modelling.

Extreme Value Theory (EVT) has been used to model the right tail of the severity of loss distributions.

The Expectation and Maximization (EM) algorithm has been applied to estimate the parameters of the frequency and severity of loss distributions when only their left truncated distributions are available.

We conclude then with a numerical application of the proposed model which aims at evaluating the risk capital for a single financial institution. To this scope we have used, as empirical observations, the OpData® dataset supplied by OpVantage®. In order to estimate the risk capital, we calculate the Value at Risk of the simulated operational loss distribution.

Keywords: operational risk, Extreme Value Theory, Value at Risk, Expected Shortfall, Monte Carlo simulation, copula functions, EM algorithm.
JEL Classification: C13, C15, C22, C24, G22, G32.

Introduction

With the regulatory spotlight on operational risk management, there has been ever increasing attention devoted to the quantification of operational risk.

The operational risk potential devastating power has been shown by many large operational losses; some of the best known operational risk incidents are the $9 billion loss of Banco National due to credit fraud in 1995, the $2.6 billion loss of Sumimoto Corporation due to unauthorized trading activity in 1996, the $1.7 billion loss and subsequent bankruptcy of Orange County due to unauthorized trading activity in 1998, the $1.3 billion trading loss causing the collapse of Barings Bank in 1995, the $0.75 billion loss of Allied Irish Bank in 2002, the loss of $2 million of Prudential Insurance of America in 2002.

The new regulatory framework in banking sector (Basel II) and the project of the new solvency regime in insurance sector (Solvency II) recognize the importance of operational risk by requiring its explicit treatment with the determination of a specific capital requirement.

There is no generally accepted definition of operational risk in the financial community. In this paper we refer to the definition proposed by the Basel Committee on Banking Supervision in 2001: *"the risk of loss resulting from inadequate or failed internal processes, people and systems or from external events"*. This definition has been adopted for the insurance sector until now.

In this categorization operational risk includes the following event types: business disruption and system failures; clients, products and business practice; damage to physical assets; employment practice and workplace safety; execution delivery and process; external fraud; internal fraud.

In this paper, we develop a comprehensive model to quantify the capital charge necessary to cover the operational risk in a financial institution.

The proposed model belongs to the class of the "Loss Distribution Approach" (LDA). LDA is a frequency/severity model widely used in many fields of the actuarial practice and comes from Basel II methodology framework for operational risk estimation (advanced models).

In this model the frequency and severity of loss distributions are determined for each of the events of loss identifying the best fitting distribution of empirical data (see Moscadelli, 2004; De Fontnouvelle, 2003, 2004); then we apply copula functions to reflect the dependence amongst the different events dealing with operational losses (see, for example, Di Clemente-Romano, 2003; Reshetar, 2004). The operational risk capital charge is estimated by quantifying the Value at Risk and Expected Shortfall of the joint distribution of

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losses (see Rockafellar-Uryasev, 2000; 2002), estimated using Monte Carlo simulation.

In order to better estimate the possibly fat tails of the severity distributions, the Extreme Value Theory (Embrechts et al., 1997, 2001, 2003, 2004, 2005) is also included in model.

The paper includes an application aimed at the evaluation of operational risk capital charge.

The numerical example has been carried out using the OpData dataset supplied by OpVantage that is the largest dataset existing on the market concerning operational losses.

In the OpData dataset only operational risk losses exceeding $1 million are collected. This limitation may be overcome using appropriate algorithms to solve the problem of estimating the severity/frequency distribution with truncated data. In our model we adopt the EM algorithm (see Bee, 2005; Chernobai, 2006).

The main tools we proposed in this paper (copula functions, EM algorithm and Extreme Value Theory) are widely used in financial and actuarial practice. Nevertheless, we used them simultaneously in order to capture all the features of operational risk modelization.

The paper has the following structure:

Section 1 is devoted to the methodology overview; here we describe our model of evaluation. In particular, section 1.1 deals with the Loss Distribution Approach and explains the structure of a typical model of this family; introduces copula functions to model dependencies amongst different risk types, gives a brief overview of the risk measures (VaR and Expected Shortfall) chosen to quantify the capital charge needed to cover the operational loss risk and introduces the Extreme Value Theory.

Section 1.2 is dedicated to the distribution estimation with truncated data and presents the EM algorithm that is used in the following section 2.

Section 2 illustrates a numerical application. The model is developed using the OpData dataset as input data. The methodology procedures and algorithms described in section 1 are applied on empirical data to test the model application. The results represent the quantification of the operational risk capital charge in a hypothetical financial institution.

1. Model description

In this section we describe the "technical blocks" that compose our model.

In particular, we present the main characteristics of the Loss Distribution Approach (LDA), some useful risk measures to quantify the operational risk capital charge and some basic elements about copula functions and Extreme Value Theory; we also introduce the EM algorithm that is a useful tool to manage truncated datasets.

1.1. An overview of Loss Distribution Approach, copula function and Extreme Value Theory (EVT). As already said in the introduction, our model may be considered a Loss Distribution Approach (LDA) for the quantification of operational risk.

To apply the Loss Distribution Approach we need to determine the loss frequency and the loss severity distributions for each event type. The aggregate distribution for each event type is then obtained as the convolution of the frequency and the severity of loss distributions.

Let us denote N_i the loss frequency for event type i, and Y_{ik} the loss severity associated to the *k-th* event for event type i and with F the probability distribution of the aggregate losses.

In the Loss Distribution Approach framework, the aggregate annual loss Y_i, for event type i, can be obtained as the sum of the stochastic number N_i of events occurred in one year with severity Y_{ik}:

$$Y_i = \sum_{k=0}^{N_i} Y_{ik} \quad . \tag{1}$$

We assume the following:

- losses Y_{ik} are i.i.d. random variables;
- loss frequencies and loss severities are independent random variables.

The analytical expression for $G_i(y)$ cannot be derived in general. Therefore, we must employ a Monte Carlo simulation in order to generate a high number of simulated aggregated losses.

In order to correctly apply tail risk measures such as Value at Risk (VaR) or Expected Shortfall (ES), we must efficiently estimate the tails of severity distributions. Indeed, severities data have generally heavy tailed distributions. In this situation, Extreme Value Theory (EVT) permits to take into account large losses in a correct way.

We recall here that if we fix a confidence level $\alpha \in (0,1)$, the Value at Risk (VaR) and the Expected Shortfall (ES) for the loss random variable X at probability level α with cdf $F_X(x)$ is defined as:

$$VaR_a = min\{\varsigma \in R : F_X(\varsigma) \geq \alpha\}, \tag{2}$$

$$\mathrm{ES}_\alpha = E\left[X \mid X > VaR\right]. \tag{3}$$

In the first definition, the VaR_α turns out to be the left endpoint of the nonempty interval consisting of the values ς such that $F_X(\varsigma) = \alpha$. Besides we

deduce that the probability that $X \geq \mathrm{VaR}_\alpha$ equals $1-\alpha$. Consequently, the ES_α is defined as the conditional expectation of the losses which are greater to VaR_α.

In this model we also use the EVT approach which is useful to correctly estimate heavy tails distributions.

EVT aims at describing the distributions of rare events by focusing on the tail of the distribution.

EVT is then a powerful tool for managing losses due to rare events and inadequacy of internal controls (Low Frequency High Impact Events).

In general, extreme events can be treated in the two following ways:

- by considering the maximum (or the minimum) of the analyzed random variable in consecutive periods (for example months or years) in which the time horizon has been subdivided (*"Block Maxima"* method);
- by considering the value that the random variable assumes over a given threshold ("*Peaks Over Threshold method", POT*).

In the rest of the paper, we will consider exclusively the *POT* method.

Traditionally, the total capital charge for operational risk is obtained by summing capital charges for each event type. This procedure implies a perfect dependence between each event type (in other words, aggregate losses coming from different event types are comonotonic random variables). This unrealistic assumption implies the overestimation of the total capital charge. The utilization of a more realistic dependence structure, through copula functions, produces a lower capital charge (diversification effect). To this aim we choose the t-Student Copula.

When we consider more than two marginal distributions, the Archimedean family is no more efficient because this copula has only one parameter (thus the same dependence structure between each couple of marginal distributions is assumed). The t-Student Copula is then more appropriate as the tail dependence is captured (through the degrees of freedom). For these reasons, t-Student Copula is widely used in literature.

The main characteristics of the t-Student Copula are here briefly described.

Its parameters are the correlation matrix R and the degrees of freedom υ.

Let X be a vector with a $n-$variate standardized Student's t-distribution with υ degrees of freedom, and covariance matrix $\frac{\upsilon}{\upsilon-2}R$ (for $\upsilon>2$). It can be defined in the following way:

$$X := \frac{\sqrt{\upsilon}}{\sqrt{S}}Y \tag{4}$$

where $S \sim \chi^2_\upsilon$ (the chi-square distribution) and the random vector $Y \sim N_n(0,R)$ are independent.

The copula of vector Y is the t-Student Copula with υ degrees of freedom. The analytical representation is the following:

$$C^n_{\upsilon,R}(u_1,\ldots,u_n) = t^n_{\upsilon,R}\left(t^{-1}_\upsilon(u_1),\ldots,t^{-1}_\upsilon(u_n)\right) \tag{5}$$

or equivalently:

$$C^n_{\upsilon,R}(u_1,\ldots,u_n) = \int_{-\infty}^{u_1}\int_{-\infty}^{u_2}\cdots\int_{-\infty}^{u_n}\frac{\Gamma\left(\frac{\upsilon+n}{2}\right)\cdot|R|^{\frac{1}{2}}}{\Gamma\left(\frac{\upsilon}{2}\right)\cdot(\upsilon\cdot\pi)^{\frac{n}{2}}}\cdot$$

$$\cdot\left(1+\frac{1}{\upsilon}u^T\cdot R^{-1}\cdot u\right)du_1\,du_2\ldots du_n, \tag{6}$$

where $R_{ij} = \frac{\Sigma_{ij}}{\sqrt{\Sigma_{ii}\cdot\Sigma_{jj}}}$ for $i,j\in\{1,\cdots,n\}$ are the correlations. We also indicate with $t^n_{\upsilon,R}$ the multivariate cumulative distribution function of the random vector $\frac{\sqrt{\upsilon}\cdot Y}{\sqrt{S}}$, where the random variable $S\sim\chi^2_\upsilon$ and the random vector Y are independent. Besides, t_υ (cdf of the standard univariate Student distribution) denotes the margins of $t^n_{\upsilon,R}$.

Finally, the t-Student Copula has the following density:

$$c(u_1,\ldots,u_n;R,\upsilon) = \frac{\Gamma\left(\frac{\upsilon+n}{2}\right)\cdot\left[\Gamma\left(\frac{\upsilon}{2}\right)\right]^n\cdot\left(1+\frac{1}{\upsilon}\omega^T\cdot R^{-1}\cdot\omega\right)^{-\frac{\upsilon+n}{2}}}{\sqrt{|R|}\cdot\Gamma\left(\frac{\upsilon}{2}\right)\cdot\left[\Gamma\left(\frac{\upsilon+1}{2}\right)\right]^n\cdot\prod_{i=1}^n\left(1+\frac{\omega_i^2}{\upsilon}\right)^{-\frac{\upsilon+1}{2}}}, \tag{7}$$

where $\omega_i = t_\upsilon^{-1}(u_i)$.

If we choose a t-Student Copula, the degree of freedom υ can be evaluated with a log likelihood estimator.

Once the marginal distributions have been chosen, this can be accomplished through a Maximum Likelihood Estimation (MLE).

To generate pseudo-random numbers from the t-Student Copula we use the following algorithm:

- find the Cholesky decomposition *A* of the correlation matrix *R*;
- simulate *n* independent random variates $z = (z_1, \ldots, z_n)$ from the standard normal distribution;
- simulate a random variate *s* from χ^2_υ distribution, independent of *z*;
- determine the vector $y = A \times z$;
- set $x = \frac{\sqrt{\upsilon}}{\sqrt{s}} y$;
- determine the components $u_i = t_\upsilon(x_i) \quad i = 1, \ldots, n$. The resultant vector is $(u_1, \ldots, u_n)^T \propto C^n_{\upsilon,R}$.

An exhaustive description of copula functions and related algorithms described above can be found in Mashal & Naldi (2002) and Di Clemente & Romano (2003).

1.2. EM algorithm. In order to face the truncation data problem, a large amount of tools are available in the literature (such as EM algorithm or techniques that use the truncated Maximum Likelihood estimator). In this paper, we choose the EM algorithm. This methodology, applied to the lognormal distribution, appeared first in a paper of Chernobai et al.. (A note on the estimation of the frequency and severity distribution of operational losses, 2006) and can be also found in subsequent papers (Bee, 2005).

The model is designed keeping in mind the availability of truncated data. In the operational risk losses databases we can find in the markets the lower losses are generally not reported. For internal databases the threshold is generally $ 10,000 while for external databases this threshold is generally $ 1,000,000.

The problem of determining the parameters of the distribution which represents empirical loss frequencies and loss severities is obviously influenced by the presence of "truncated data".

If the database consists of truncated data the traditional best fitting techniques based on maximum likelihood estimation or percentile matching, cannot be applied. In this case it is necessary to apply some specific tools such as the so called EM algorithm to evaluate the parameters of the unknown complete distribution.

To this aim, the conditional distributions and maximum likelihood estimation techniques involving truncated data are considered.

We assume the standard hypothesis that the severity distribution follows a lognormal distribution. Some recent empirical studies showed that severity distributions can be also modelled by $\alpha-$stable distributions (Rachev et al., 2007) or "*g-and-h*" distributions (Embrechts et al., 2006; Dutta&Perry, 2007). These distributions are suitable for estimating heavy tails.

The maximum likelihood estimation in presence of incomplete data can be carried out using the EM algorithm (Expectation-Maximization) (see Dempster et al., 1977).

The estimation of the parameters of the complete distribution is done by maximizing the *loglikelihood* function (see Frachot et al., 2003).

Let $Y = (N_{NT}, y)$ be the observed incomplete sample, where N_{NT} denotes the number of observed data and $y = (y_1, y_2, \ldots, y_{N_{NT}})$ is the observed data vector. We also denote by $X = (N_T, x)$ the missing data, where N_T represents the number of missing data and $x = (x_1, x_2, \ldots, x_{N_T})$ is the vector of missing data.

The complete data will then assume the following form:

$Z = (X, N_T, Y, N_{NT})$ with $y = \{z_i : z_i \geq s\}$;

$x = \{z_i : z_i < s\}$; $N_{NT} = \#\{z_i : z_i \geq s\}$;

$N_T = \#\{z_i : z_i < s\}$ and *s* is the truncation threshold (see Bee, 2005)

Let us denote:

$l = l(\theta)$ the *loglikelihood* function for Y;

θ the parameters vector we have to estimate;

$l_C = l_C(\theta)$ the *loglikelihood* function for the complete data;

The two steps of the algorithm are:

1) E-step: estimate the expected conditional value of the *loglikelihood* function $l_C = l_C(\theta)$ for the observed sample *y* and the current value for θ.

Let us denote θ_0 the initial value of the vector parameters (determined arbitrarily), we must then estimate:

$$Q(\theta;\theta_0) = E_{\theta_0}\left\{l_C(\theta)\middle|y\right\}. \quad (8)$$

2) M-step: the previous expression must be maximized with respect to θ:

$$Q(\theta_1;\theta_0) = \max Q(\theta;\theta_0). \quad (9)$$

We repeat the procedure by substituting θ_0 with θ_1.

These two steps are then repeated until convergence occurs.

We describe now the EM algorithm in the hypothesis of lognormal distribution.

Given a sample $w_1, w_2, \ldots, w_{N_{NT}}$ coming from a lognormal truncated distribution with parameters μ and σ^2 (N_{NT} is the number of observed data) we determine $y_i = \log(w_i) \approx N(\mu,\sigma^2)$ in order to transform the data into a normal distribution.

The complete distribution function is:

$$L_C(\mu,\sigma^2;z) = \prod_{i=1}^{N} L(\mu,\sigma^2;z_i), \quad (10)$$

where $\mu \in R$, $\sigma^2 \in R^+$ and $L(\mu,\sigma^2;z_i)$ is the normal distribution function estimated in z_i.

The observed distribution function is:

$$L_{obs}(\mu,\sigma^2;s,y) = \prod_{i=1}^{N_{NT}} L_{obs}(\mu,\sigma^2;s,y_i) = \prod_{i=1}^{N_{NT}} \left\{ \frac{1}{1-\Phi(s^*)} \cdot \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{y_i-\mu}{\sigma}\right)^2} \right\} \quad (11)$$

with $\mu \in R$, $\sigma^2 \in R^+$, $s^* = \dfrac{s-\mu}{\sigma}$.

In the implementation of EM algorithm, for calculating:

$$L_C(\mu,\sigma^2;z) = \prod_{i=1}^{N} L(\mu,\sigma^2;z_i). \quad (12)$$

we use the fact that the joint distribution of Y, N_T and X can be expressed in the following way:

$$f\left(y, N_T, x\right) = f\left(y\right) \cdot f\left(N_T \middle| y\right) \cdot f\left(x \middle| N_T, y\right)$$

As the function $f(y)$ is already known, it remains to determine $N_T\,|\,y$ and $x\,|\,N_T, y$.

Given the complete data set Z from a normal distribution with parameters μ and σ^2, it has been proved (McLachlan, Krishnan, 1996) that $N_T\,|\,y$ follows a negative binomial distribution with parameters N_{NT} and $K = \Pr(Z > s) = 1 - \Phi(s^*)$.

Consequently, the expected conditional value is:

$$E\left(N_T \middle| y\right) = N_{NT} \cdot \frac{1-K}{K}. \quad (13)$$

The distribution of the missing data X is a right truncated normal distribution with density function:

$$f_X(x) = \frac{1}{\Phi(s^*)} \cdot \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, \; x < s. \quad (14)$$

The estimation of $E\left[L\left(\mu,\sigma^2;s,x_i\middle|y_i,\mu^{(t)},\sigma^{2(t)}\right)\right]$ is rather easy due to the fact that the loglikelihood function is linear in X and X^2, so that we have to estimate:

$$E\left(X\middle|\mu^{(t)},\sigma^{2(t)},y_1,y_2,\ldots,y_{N_{NT}}\right), \quad (15)$$

$$E\left(X^2\middle|\mu^{(t)},\sigma^{2(t)},y_1,y_2,\ldots,y_{N_{NT}}\right). \quad (16)$$

The estimation of (15) and (16) can be performed thanks to the known formulas for the expected value and the variance of the right truncated normal distribution.

We set $\alpha\left(s^*\right) = \dfrac{\phi\left(s^*\right)}{\Phi\left(s^*\right)}$ so that:

$$E\left(X\middle|\theta^{(t)},y\right) = \mu^{(t)} - \sigma^{(t)} \cdot \alpha\left(s^{*(t)}\right), \quad (17)$$

$$E\left(X^2\middle|\theta^{(t)},y\right) = \sigma^{2(t)} \cdot \left[1 - s^{*(t)} \cdot \alpha\left(s^{*(t)}\right) - \left(\alpha\left(s^{*(t)}\right)\right)^2\right] + \left[E\left(X\middle|\theta^{(t)},y\right)\right]^2. \quad (18)$$

Finally, the E-step is given by:

$$E\left(N_T \mid y\right) = \hat{N}_T = N_{NT} \cdot \frac{1-K}{K}$$

$$E\left(X\left|\theta^{(t)},y\right.\right)=\mu^{(t)}-\sigma^{(t)}\cdot\alpha\left(s^{*(t)}\right)$$

$$E\left(X^2\left|\theta^{(t)},y\right.\right)=\sigma^{2(t)}\cdot\left[1-s^{*(t)}\cdot\alpha\left(s^{*(t)}\right)-\left(\alpha\left(s^{*(t)}\right)\right)^2\right]+\left[E\left(X\left|\theta^{(t)},y\right.\right)\right]^2$$

Now the M-step requires the formulas for the maximum likelihood, in the case of normal distribution, for the missing data coming from (17) and (18):

$$\mu^{(t+1)}=\frac{1}{N_{NT}+N_T^{(t)}}\left(\sum_{i=1}^{N_{NT}}y_i+N_T^{(t)}\cdot E\left(X\left|\theta^{(t)},y\right.\right)\right), \tag{19}$$

$$\sigma^{2(t+1)}=\frac{1}{N_{NT}+N_T^{(t)}}\left(\sum_{i=1}^{N_{NT}}y_i^2+N_T^{(t)}\cdot E\left(X^2\left|\theta^{(t)},y\right.\right)\right)-\left(\mu^{(t+1)}\right)^2. \tag{20}$$

The parameters are then obtained by iterating (17), (18), (19) and (20) until convergence.

2. A model application

An application of the model described above has been developed with the aim to calculate the operational risk capital charge in a financial institution (bank or insurance company).

The input data are derived from OpData, an operational losses database supplied by OpVantage, a division of Fitch Risk Management.

The data are collected from public sources and in the database only losses, whose amounts exceed a truncation threshold of $ 1 million, are registered during the period of 1972-2006.

In this application we consider the database for the period of 1994-2006, since from 1972 to 1994 it is not statistically significant. The database contains worldwide firms and we selected a sub-sample corresponding to the sector "financial services" with firms whose total assets are reported. With these restrictions, the sub-sample we used contains 1.025 records.

In the OpData, operational losses are categorized according to the Basel Committee's event types classification:

1) business disruption and system failures;
2) clients, products and business practice;
3) damage to physical assets;
4) employment practice and workplace safety;
5) execution delivery and process management;
6) external fraud;
7) internal fraud.

Due to the lack of the loss data in event types 1) and 3), we have taken into account five event types' data.

For each loss the following information is available:

- classification by event type;
- firm name;
- loss event description;
- loss amount in local currency;
- loss amount in dollars;
- loss amount in current value dollars (based on CPI);
- loss data;
- country;
- total assets of the firm.

The model consists of the following steps:

- estimation of the parameters of frequency and severity distributions, for each event type by applying the EM algorithm. The frequency of loss arising from each event type is assumed to be a Poisson distribution while the lognormal distribution is used to model the severity of loss;
- estimation of the aggregate loss distribution, for each event type, via Monte Carlo simulation;
- quantification of operational risk capital charge, for each event type, through risk measure as Value at Risk and Expected Shortfall;
- quantification of the total operational risk capital charge under different hypotheses:
  - perfect dependence (comonotonicity) among event types. The total capital charge is then obtained by summing capital charges for each event type and it results overestimated;
  - independence between event types. This assumption leads to underestimation of the total capital charge;
  - realistic dependence structure through a t-Student copula.

For efficiently modelling the right tail of the severity distribution we repeat the above steps using Extreme Value Theory (Di Clemente-Romano, 2003; Embrechts, 2003, 2005; Moscadelli, 2004). Therefore we model the severity distribution using the lognormal distribution (in the left tail and in the centre) and the Generalized Pareto Distribution (GPD) for the right tail.

The database at our disposal is structured as follows: for each year we know the determination k of the

random variable N "number of loss events in one year" and consequently we have at our disposal k determinations of the random variable Y_i.

Then, for each financial institution, we normalize its loss amount y_{ik} by dividing it by the total asset A: $\hat{y}_{ik} = \frac{y_{ik}}{A}$; in this way we shall obtain results that are expressed as percentages of the total assets of the firm.

We apply then the EM algorithm to estimate the parameters of frequency and severity distributions (see Table 1).

Table 1. Parameters estimation of severity and frequency distributions for each event type

Event type	Severity (lognormal)	Frequency (Poisson)
1. Clients, products and business practices	$\overline{\mu} = -10.425$; $\overline{\sigma} = 2.286$	$\lambda = 37.130$
2. Employment practices and workplace safety	$\overline{\mu} = -12.139$; $\overline{\sigma} = 2.066$	$\lambda = 6.686$
3. Execution, delivery and process management	$\overline{\mu} = -11.456$; $\overline{\sigma} = 2.039$	$\lambda = 6.678$
4. External fraud	$\overline{\mu} = -10.824$; $\overline{\sigma} = 1.975$	$\lambda = 13.741$
5. Internal fraud	$\overline{\mu} = -10.692$; $\overline{\sigma} = 2.143$	$\lambda = 18.971$

The aggregate loss distribution, for each event type i, is obtained as a convolution of the frequency and severity distributions with the Monte Carlo simulation method.

We can estimate operational losses at firm level, for each event type i, by dividing aggregated losses by the number of firms that suffered from event type i.

We consider Value at Risk and Expected Shortfall as the capital amount needed to cover operational risk.

Capital charges, at different confidence levels, for each event type are reported in Tables 2 and 3 (Monte Carlo simulation with 100,000 replications). We remind that Value at Risk and Expected Shortfall estimation is reported to unitary capital loss.

Table 2. Value at risk for event type

Event type	VaR 95%	VaR 99%	VaR 99.9%
1. Clients, products and business practices	0.000564	0.001309	0.004634
2. Employment practices and workplace safety	0.000045	0.000116	0.000411
3. Execution, delivery and process management	0.000051	0.000133	0.000477
4. External fraud	0.000092	0.000210	0.000637
5. Internal fraud	0.000178	0.000431	0.001260
TOTAL	0.000929	0.002199	0.007419

Table 3. Expected shortfall for event type

Event type	ES 95%	ES 99%	ES 99.9%
1. Clients, products and business practices	0.001209	0.002871	0.009721
2. Employment practices and workplace safety	0.000103	0.000243	0.000735
3. Execution, delivery and process management	0.000121	0.000301	0.001043
4. External fraud	0.000183	0.000398	0.001127
5. Internal fraud	0.000366	0.000807	0.002224
TOTAL	0.001982	0.004620	0.014850

Finally, we model the dependence structure among event types by using a t-Student Copula.

The MLE estimation for the optimal degrees of freedom of the copula gives $\upsilon = 5$.

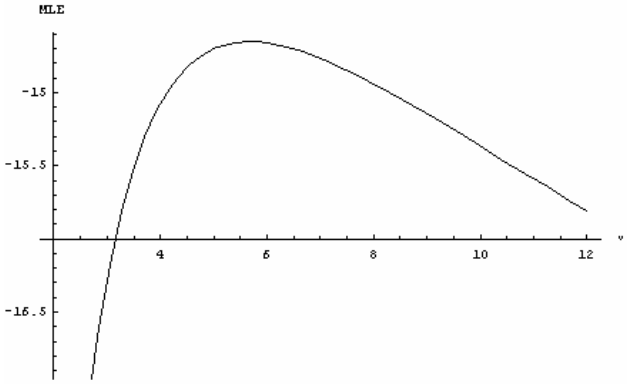

Fig. 1. The estimation of the optimal degrees of freedom

To apply the chosen copula we must estimate the correlation matrix of the risk events.

We cannot use an empirical correlation matrix (i.e., inferred from historical data) due to database truncation; for this reason, we consider a theoretical correlation matrix based on qualitative considerations.

The following table shows the correlation matrix.

Table 4. Qualitative correlation matrix (medium correlation = 0.35, high correlation = 0.55)

Qualitative correlations among events	1	2	3	4	5
1	1				
2	medium	1			
3	high	medium	1		
4	zero	zero	high	1	
5	high	zero	high	medium	1

The aggregate loss distribution with copula function is obtained through algorithms presented in section 1.1.

In Table 5 the total operational risk capital charge for the t-Student Copula dependence structure together with the Comonotonicity and Independence cases are shown.

Table 5. Operational risk capital charge (% of total asset A)

OR capital charge	VaR 95%	VaR 99%	VaR 99.9%	ES 95%	ES 99%	ES 99.9%
Comonotonicity	0.000929	0.002199	0.007419	0.001982	0.004620	0.014850
t-Student copula	0.000836	0.001932	0.005991	0.001694	0.003793	0.010647
Independence	0.000755	0.001566	0.004839	0.001437	0.003139	0.009965

As it is expected, considering the dependencies among event types in case of application of t-Student Copula, the operational risk capital charge, expressed as a percentage of the total asset, is always between the minimum values, which correspond to the assumption of independence, and the maximum values, which are obtained if perfect dependence (comonotonicity) is assumed.

The results obtained may be refined with the Extreme Value Theory, which can be used for modelling more efficiently the right tail of the severity distribution.

A right tail analysis suggests that only for the event type Internal Fraud, the lognormal distribution underestimates the large loss probability (see Figure 2).

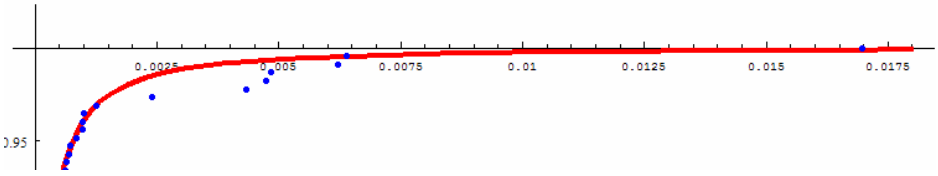

Fig. 2. Lognormal distribution (line), empirical distribution (dots)

We may "correct" the observed underestimation by modelling the right tail with the Generalized Pareto Distribution (*POT* method).

In order to fit the GPD on our data we perform the following steps:

- find the appropriate threshold k;
- determine loss excesses (loss amounts over the threshold minus the threshold);
- estimate GPD parameters ξ and σ from the excesses.

According to Figure 2, the large loss probability is underestimated by lognormal distribution over the threshold 0.0007174 (95-th percentile).

Maximum Likelihood Method has been used to estimate GPD parameters (see Table 6).

Table 6. GPD parameters internal fraud

k	ξ	σ
0.0007174	0.817819	0.001319

The best fitting between GPD (ξ, σ) and empirical data are reported in Figure 3.

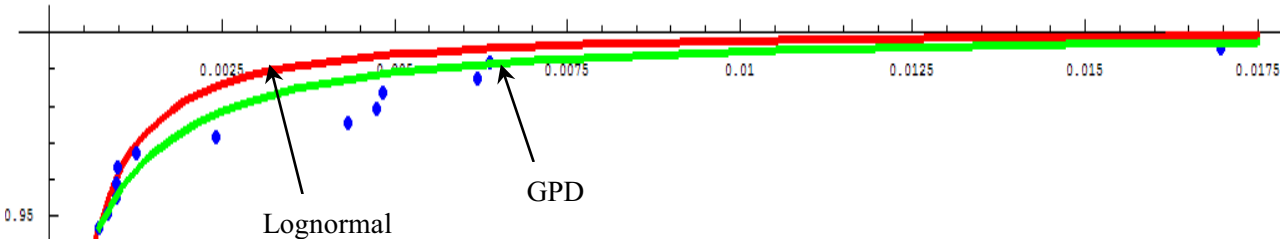

Fig. 3. Internal fraud. Lognormal distribution (dark line), empirical distribution (dots), generalized Pareto distribution (clear line)

The results are reported in Table 7.

Table 7. Operational risk capital charge with EVT (% of total asset A)

OR capital charge	VaR 95%	VaR 99%	VaR 99.9%	ES 95%	ES 99%	ES 99.9%
Comonotonicity	0.000940	0.002368	0.009648	0.002263	0.005923	0.024049
t-Student copula	0.000849	0.002132	0.008214	0.001972	0.004917	0.017879
Independence	0.000782	0.001749	0.006577	0.001684	0.004133	0.016445

Comparing the results of Table 7 with those one in Table 5, we can appreciate the importance of choosing the appropriate probability distribution to fit the right tails.

Conclusions

The application is developed using operational risk data of financial institutions, based in different countries, with different business dimension and risk profile.

For this reason we have to interpret it mainly as an exercise to check the model feasibility and to understand the difficulties that the evaluator may encounter in practice. But, at the same time, the model results allow us to emphasize the importance of considering the dependence among the events and its impact on the operational risk capital charge.

A similar comment is valid for the technical consideration of the fat tails of the severity of loss distributions.

In addition, we find that the possibility of quantifying the risk capital when only left truncated loss distributions are available is relevant from the practical point of view.

It is well known that the most common concern of risk managers is the lack of empirical data. With this model, we intend to suggest that, as first approach, the financial institutions could start to collect operational risk data over a fixed threshold.

A last consideration: we calculate (Tables 5 and 7) the operational risk capital charge as a percentage of the total asset at different confidence levels. This means that the single financial institution could measure its own operational risk capital charge simply by multiplying this percentage by its total asset.

We could think that this kind of evaluation be carried on, for example, at the country level and the results utilized as a benchmark.

To improve the performances of the LDA model proposed in this paper and to achieve a more precise monitoring of the operational risk, it is important to focus on the role played by the operational loss database that the financial institution can use in order to estimate its operational losses.

In particular, databases concerning operational loss data are quite rare and not easily traceable in the market. We highlight that a "good quality" database can increase the significance of the final results of the risk analysis.

Due to the fact that some particular operational losses are rare events (i.e., for the category low frequency/high impact events) it's obvious that a financial institution cannot use only its own data (i.e., data concerning its own operational losses) to estimate the parameters of the LDA model but it must integrate its own data with databases sold by specialized data-provider.

These databases should contain not only the relevant information about the loss events (amount, date, event type and so on) but also a detailed description of the main characteristics of the entity that suffered the operational loss like, for example, the activity sector, the size, the geographical location and the main indicators of the balance sheet at the loss event date.

If databases share this particular structure, each valuator may assess its technical bases in "tailor made" manner that is the best way to perform valuation in this field.

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