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Development under a concessionary agreement: a real option approach

Abstract

In this article we study the situation, where a private company is able to obtain a concession by the state to develop a project, and is able to return this concession at its own will, when the project becomes unprofitable. The latter may result in a fee that the company needs to pay to the state. We are particularly thinking about the development of state-owned land in China and Thailand for the matter of building private schools, hospitals, factories, roads or expressways. To model this, we use multi stage real option theory. In particular, we discuss the cases where the project value follows either a geometric mean reversion process or a geometric Brownian motion. For these cases we derive the Bellman equations and show how the problem can be solved backwards in time. The resulting free boundary problems are solved numerically via the shooting method. A comparative analysis is provided. Particular emphasis is given to the role of uncertainty and how uncertainty affects the average time that the concessionary agreement is in action. The latter problem is approached by using Monte Carlo simulation.

Keywords: investment, real options, dynamic programming.
JEL Classification: C61, G11, G12, G31.

Introduction

In countries such as China, Thailand, Australia or Russia, many sectors of the economy are or have initially been state owned. Such sectors include but are not limited to natural resources, land, infrastructures, services such as the postal service, dental care or transportation services, etc. In particular in the services sector full privatization has been taken place in many cases. For other sectors, say natural resources such as coal, oil or nuclear energy, it may not be ideal for the state to proceed to full privatization. In the cases mentioned, this may well be as the states souvereignity or security is endangered. On the other side, private companies may have developed great expertise and efficiency to undertake projects in specific sectors that are relevant for the state-economy. Further, the state budget may not allow sufficient investment to develop each and every part of the economy. For example, the government of Thailand has had many construction plans for building inter-city roads and expressways in order to reduce the traffic congestion in the city centres. The government, however, has been unable to undertake all projects mostly due to financial constraints. In these cases it may be worthwhile for the state government to give, a concession to a private company to develop a specific project until either the concession ends at a given date in the future, or the private company ends the concession premature, paying an appropriate penalty fee to the government. When the concession has ended, all property rights fall back to the state and a new concession can be arranged. In this article we study the situation, where the concession will not be given to the same private company again, if this company decides to end the concessionary agreement prematurely. This assumption is realistic in many cases, where private companies compete for state concessions.

Entry-exit models in the real option context have been discussed in the literature (see, for example, Mossin (1968), Brennan and Schwartz (1985), Dixit (1989) and Sodal (2006)). There, private companies retain whatever market power they had to start with, and do not lose the right to invest again if they abandon operation. Our model is different from these, in the way that we do not allow for re-entry, which is realistic under the assumptions indicated above. This makes the problem essentially much harder to solve. Also unique is our use of exponential utility for modeling the benefits of the private company when undertaking the project.

Mathematically we model the level of development $x(t)$ of the project in two different ways. In the first instance we assume that the level follows a geometric mean reverting process, while in the second instance we assume it follows a geometric Brownian motion. From the point of dynamics, these are standard assumptions in real option theory. Technically, geometric mean reversion models the case where the level is mean reverting in a way that in the long term it keeps fluctuating around a so called mean reversion level θ. In this case the level features bounded variance and in expectation will converge to a certain value, which is the mean reversion level minus some expectation bias (see Ewald and Yang (2007)). This is a good modeling assumption in the case of renewable resources. Geometric Brownian motion, on the other side, grows exponentially in expectation and its variance will eventually become arbitrary large. Both, geometric mean reversion and geometric Brownian motion have

in common that if hypothetically the level of $x(t)$ would fall to zero, it would stay there forever. This has implications on the value function, which have in part been discussed in Ewald and Wang (2010), where as an alternative the Cox-Ingersoll-Ross process is proposed for modeling in the real option context.

The level of development $x(t)$ does not necessarily have to be a monetary value, it could be, for example, the level of industrialization, or the percentage of households that have access to medical treatment in the form of modern hospitals. During the time a private company develops the project under the state concession, the private company will accumulate utility measured in terms of a utility function $F_1(x(t))$, which measures benefits against costs.

Most of the classical real option models in the literature, including aforementioned authors, do not allow for a fully analytic solution. In the classical cases, it is possible to compute the solution of the dynamic programming equation analytically, but the threshold level for investment needs to be computed numerically. In our model we have to go one step further. Due to the fact that we do not allow for re-entry and the fact that we use a more complex utility function, we are not able to solve the dynamic programming equation analytically, at least not for the part that corresponds to the period in which the private company is developing the project. Instead we use a numerical method called "shooting method" to deal numerically with the corresponding free boundary problem. A detailed discussion of the numerical results, including a thorough comparative analysis, is provided.

The relationship between uncertainty and the expected time the private company develops the project under the concessionary agreement is an interesting aspect for both, the state and the private company. In other real option models the relationship between uncertainty and investment has been frequently discussed. Authors such as McDonald and Siegel (1986), Dixit (1989), Mauer and Ott (1995) as well as Metcalf and Hassett (1995) find that a rise in uncertainty leads to a larger critical value as the real option increases in price and it becomes more profitable to hold on to the option. Carr, Ewald and Xiao (2008) as well as Ewald and Yang (2008) provide examples that in case the underlying dynamic is more complex or in case that risk aversion is taken into account in a genuinely incomplete setup, these results do not necessarily hold in a more general context of real option theory. Furthermore Sarkar (2000) argues that the relationship between uncertainty and investment is not necessarily expressed in the relationship between volatility and threshold, as an increase in volatility may force the level to reach the threshold earlier rather than later, even though the threshold is higher, time until investment is undertaken is shorter. Instead, Sarkar proposes to study the expected time that passes until investment is undertaken in terms of the level of uncertainty. We will discuss both of these aspects in this article. As opposed to Sarkar (2000) who studies a single investment problem, we are facing two times, τ_1, when the private company enters the concessionary agreement, and τ_2, when the private company ends it. Our focus will be on the expected time $\mathsf{E}(\tau_2 - \tau_1)$ that the private company is developing the project.

The remainder of the paper is structured as follows. In section 1 we will set up our single entry-exit model while in section 2 we will discuss how to solve it. Section 3 is devoted to numerical results and their discussion, this includes a detailed comparative analysis. In the last section we summarize conclusions. The article contains two appendices, one which contains all figures and one in which we briefly illustrate the shooting method that is at the center of our numerical analysis.

1. Model setup

As indicated earlier, we study the situation where a project is in the first instance developed by the state, until a private company takes over under a concessionary agreement, which it can end at any given time under payment of a penalty fee. The level of the development is denoted with $x(t)$. The two main questions for the private company are: When is the optimal time to enter the concessionary agreement, and When is it optimal to end it? This will naturally lead us to a two stages real option problem which we will set up below.

1.1. The case of geometric mean reversion. In this section, we describe the model where the level of development follows geometric mean reversion. In this case the dynamic optimization problem of the private company is given as

$$V(x) = \max_{\tau_1 < \tau_2} \mathsf{E}\left[\left(\int_{\tau_1}^{\tau_2} e^{-rt} F_1(x(t))dt + e^{-r\tau_2}\left(S(x(\tau_2)) - I_q\right)\right) - e^{-r\tau_1} I_s \,\middle|\, x(0) = x\right], \quad (1)$$

s.t. $dx(t) = (\alpha - \delta x(t))x(t)dt + \sigma x(t)dW(t); \quad t \le \tau_1,$ (2)

$$dx(t) = (\mu - \delta x(t))x(t)dt + \sigma x(t)dW(t); \quad \tau_1 \le t \le \tau_2. \quad (3)$$

The interpretation of this is as follows. Before the company invests in the project at time τ_1 the dynamics of the level of the project, $dx(t)$, follows (2). During that period the development is entirely undertaken by the state government. The parameter α represents the government contribution to the project and δ is the depreciation rate. We assume $\alpha > \delta > 0$. The parameter $\sigma > 0$ denotes the volatility and $W(t)$ stands for a standard Wiener process. Once the private company decides to enter the concessionary agreement at time τ_1, the company needs to pay a fee $I_s > 0$ to the government. In addition, the dynamics of the development level changes due to the private company now running the business. We assume that $x(t)$ then follows the dynamics (3) until time τ_2 at which the private company decides to end the concessionary agreement. At that time the dynamics would go back to (2), but as the private company does not earn and in fact will never again earn any benefits from the project, this fact does not contribute to the private companies' optimization problem. Between time τ_1 and τ_2 the parameter μ represents the private companies' contribution to the project and $F_1(x(t))$ stands for the utility that the private company obtains from the development. At the time the private company decides to end the concessionary agreement, it will give the initial capital back to the government, but will be reimbursed $S(x(\tau_2))$ for the development undertaken, e.g. factory buildings that have been erected on state owned land etc. Additionally, the private company will have to pay a penalty fee of I_q.

We will solve the dynamic optimization problem (1)-(3) backward in time as it is standard in multi-stage real option problems. Let $V_0(x)$ and $V_1(x)$ denote the value functions of the corresponding dynamic optimization problems before τ_1 and in between τ_1 and τ_2. After the private company ends the concessionary agreement it will obtain a terminal payoff of $S(x(\tau_2)) - I_q$, but there will be no option value left. Note that $V(x) = V_0(x)$ and this function includes the combined option value, the one to enter and the one to exit, while $V_1(x)$ only includes the option value of exiting. It follows from standard real option theory that the private company will enter the concessionary agreement when a certain investment threshold x_s^* is reached. At this threshold the so called value matching condition (4) and smooth pasting condition (5) need to be satisfied. Further, the company will end the concessionary agreement when a second investment threshold x_q^* is reached at which the value matching condition (6) and smooth pasting condition (7) apply.

$$V_0(x_s^*) = V_1(x_s^*) - I_s, \tag{4}$$

$$\left(V_0(x_s^*)\right)' = \left(V_1(x_s^*)\right)', \tag{5}$$

$$V_1(x_q^*) = S(x_q^*) - I_q, \tag{6}$$

$$\left(V_1(x_q^*)\right)' = \left(S(x_q^*)\right)'. \tag{7}$$

1.2. The case of geometric Brownian motion. As an alternative to the setup based on geometric mean reversion we consider the following setup, which is geometric Brownian motion based. The private company aims to maximize (1) subject to

$$dx(t) = \alpha x(t)dt + \sigma x(t)dW(t); \quad t \le \tau_1, \tag{8}$$

$$dx(t) = \mu x(t)dt + \sigma x(t)dW(t); \quad \tau_1 \le t \le \tau_2. \tag{9}$$

The parameters α, μ and σ are interpreted in the same way as before. Note that there is no depreciation here, and in fact (8) and (9) are special cases of (2) and (3) for $\delta = 0$. The analysis, however, is different from the case of $\delta \neq 0$, that is why we include it as a separate case here.

2. Solving the problem

In this section we derive the partial differential equations that will determine the solutions of the optimization problems (1)-(3) resp. (1), (8)-(9) and discuss how to solve them with a combination of numerical and analytical methods. As indicated before, we proceed backward in time. In the first step we have to find $V_1(x)$ and x_q^*. Once $V_1(x)$ is determined, we will solve for $V_0(x)$ and x_s^*.

In our model, we use exponential utility to measure the benefits that accrue with the level of the development $x(t)$ as well as linear costs, yielding to a utility function of the type $F_1(x(t)) = \left(1 - \exp(-\lambda x(t))\right) - \mu x(t)$. For simplicity we assume that the terminal payoff for the private company is given as $S(x(t)) = x(t)$. Note that the final payoff also includes the level of development that has been existing before a concessionary agreement has been set up, and that the private company would normally only be remunerated for the additional level of development that it has contributed. However, the initial level can be taken care of in the fees I_s so that in this case the payoff $S(x(t)) = x(t)$ still makes sense.

2.1. Geometric mean reversion. The equations (2) and (3) are of the geometric mean reversion type which is well understood. It is known that the equilibrium distribution of this process is a Gamma-distribution and all of its moments have been derived in Ewald and Yang (2007) for example. An analytic expression for the non-equilibrium distribution has been derived in Yang and Ewald (2010). The dynamics of (2) and (3) are tied to the mean reversion level $\frac{\alpha}{\delta}$ and $\frac{\mu}{\delta}$ respectively. The parameter δ captures how fast the value of $x(t)$ reacts to the disturbance from the mean level.

We start with solving the problem for $V_1(x)$ and x_q^*, $\tau_1 \le t \le \tau_2$ by using the constraint (3) with the two free boundary conditions (6) and (7). Using the specific forms for $F_1(x(t))$ and $S(x(t))$, the corresponding Bellman equation becomes

$$rV_1 = (\mu - \delta x)xV_1' + \frac{1}{2}\sigma^2 x^2 V_1'' + 1 - exp(-\lambda x)) - \mu x \quad (10)$$

subject to the following two conditions:

$$V_1(x_q^*) = x_q^* - I_q, \quad (11)$$

$$\left(V_1(x_q^*)\right)' = 1. \quad (12)$$

We will solve (10) subject to (11) and (12) numerically. In order to do that, an additional boundary condition is needed. This condition, which applies for both geometric mean reversion and geometric Brownian motion, comes from the fact that $x(t) = 0$ is a fixed point of the dynamics (2), (3), (8) and (9). It states

$$V_1(0) = 0. \quad (13)$$

It has been shown in Ewald and Wang (2010) that this condition is equivalent to the seemingly lessrestrictive condition $V(0) < \infty$. The idea for solving the free boundary problem numerically is as follows. We first make a guess that the optimal threshold x_q^* is $\hat{x}$, and then apply the shooting method (see appendix for details) to solve the boundary value problem (10) subject to (13) and (11). We then check whether the solution $\hat{x}$ satisfies (12) within a certain level of tolerance, which we allow for. If $\hat{x}$ does not satisfy (12), we change $\hat{x}$ to $\hat{x} + \varepsilon$ where ε is a small number. This procedure is repeated until an approximation to the solution x_q^* is found.

Once we have $V_1(x)$, we will then solve for $V_0(x)$ and x_s^*, by using the constraint (2) with the two free boundary conditions (4) and (5). The Bellman equation for this problem is

$$rV_0 = (\alpha - \delta x)xV_0' + \frac{1}{2}\sigma^2 x^2 V_0''. \quad (14)$$

This problem is essentially the same as the classical real option problem discussed in Dixit and Pindyck (1994), except that the value matching conditions are coming from $V_1(x)$, which makes it more complicated. Nevertheless the way to derive the general form of the value function before τ_1 is in complete analogy to Dixit and Pindyck. To make this article as self-contained as possible, we include this derivation here.

It is not difficult to see that an elementary solution of (14) is given by

$$V_0(x) = Ax^\beta h(x).$$

By substitution in (14) and after rearranging terms, we obtain

$$x^\beta h(x)\left[\frac{1}{2}\sigma^2\beta(\beta-1) + \beta\alpha - r\right] + x^{\beta+1}\left[\frac{1}{2}\sigma^2 x h''(x) + (\alpha - \delta x + \sigma^2\beta)h'(x) - \beta\delta h(x)\right] = 0. \quad (15)$$

Equation (15) needs to hold for all values of x, therefore the coefficients for both $x^\beta h(x)$ and $x^{\beta+1}$ must be equal to zero. From the first term of (15), we get

$\frac{1}{2}\sigma^2\beta(\beta-1) + \beta\alpha - r = 0$ which has two roots:

$$\beta_1 = \frac{-(\alpha - \frac{1}{2}\sigma^2) + \sqrt{(\alpha - \frac{1}{2}\sigma^2)^2 + 2\sigma^2 r}}{\sigma^2}, \quad (16)$$

$$\beta_2 = \frac{-(\alpha - \frac{1}{2}\sigma^2) - \sqrt{(\alpha - \frac{1}{2}\sigma^2)^2 + 2\sigma^2 r}}{\sigma^2}. \quad (17)$$

This leads to

$$V_0(x) = A_1 x^{\beta_1} h_1(x) + A_2 x^{\beta_2} h_2(x)$$

with $h_i(x); i = 1,2$ satisfying

$$\frac{1}{2}\sigma^2 x h_i''(x) + (\alpha - \delta x + \sigma^2\beta_i)h_i'(x) - \beta_i\delta h_i(x). \quad (18)$$

Substituting $z = \frac{2\delta x}{\sigma^2}$ and $h_i(x) = m_i(z)$, we obtain $h_i'(x) = \left(\frac{2\delta}{\sigma^2}\right) m_i'(z)$ and $h_i''(x) = \left(\frac{2\delta}{\sigma^2}\right)^2 m_i''(z)$. The equation (18) then becomes

$$z m_i''(z) + (b - z) m_i'(z) - \beta_i m_i(z) = 0, \quad (19)$$

where $b = \frac{2\alpha}{\sigma^2} + 2\beta.$

Equation (19) is known as the Kummer equation and its solution is given by the Kummer's M functions, denoted as M in the following. The Kummer M function is also known as confluent hyper geometric function. Details on the Kummer M function can be obtained from Abramovitz and Stegun (1972). As β_2 is negative and the Kummer M function takes the value 1 for the argument $x = 0$ we therefore need to impose the condition $A_2 = 0$ so as to get a finite value of the project at $x = 0$. The solution of (14) is therefore

$$V_0(x) = A x^{\beta} M\left(\beta, \frac{2\alpha}{\sigma^2} + 2\beta, \frac{2\delta}{\sigma^2} x\right), \quad (20)$$

where $A = A_1$ is a constant that is yet to be determined, $\beta = \beta_1$ and $M(a, b, z)$ denotes the Kummer's M function.

The free boundary conditions (4) and (5) become

$$A x_s^{*\beta} M\left(\beta, \frac{2\alpha}{\sigma^2} + 2\beta, \frac{2\delta}{\sigma^2} x_s^*\right) = V_1(x_s^*) - I_s \quad (21)$$

and

$$A x_s^{*\beta} \frac{2\beta\delta}{2\alpha + 2\beta\sigma^2} M\left(\beta + 1, \frac{2\alpha}{\sigma^2} + 2\beta + 1, \frac{2\delta}{\sigma^2} x_s^*\right) + \beta A x_s^{*(\beta-1)} M\left(\beta, \frac{2\alpha}{\sigma^2} + 2\beta, \frac{2\delta}{\sigma^2} x_s^*\right) = \left(V_1\left(x_s^*\right)\right)'. \quad (22)$$

Because $V_1(x)$ does not exist in analytic form, we have to replace it by the corresponding finite difference

$$\left(V_1(x)\right)' \approx \frac{\left(V_1(x+h)\right) - \left(V_1(x)\right)}{h}.$$

To obtain $V_0(x)$ and x_s^* we now proceed as follows: $V_1(x)$ has been obtained in the previous step numerically and we know the value of $V_1(x)$ at each $x \in \{x_0 = 0, x_1, x_2, \ldots, x_n = x_q^*\}$. We first make a guess that $x_s^* = x_1$ and then use (21) to find out A. We then substitute x_1 and A in (22). If x_1 and the corresponding A satisfy (22) up to a certain level of tolerance; i.e. the difference between LHS and RHS of (22) is less than a given small number ε (we use $\varepsilon = 10^{-4}$ here), we take $x_s^* = x_1$. Otherwise we move to x_2 and carry on. The procedure is repeated until x_s^* is found.

2.2. Geometric Brownian motion. We are using the same procedure as in the previous subsection. We start with solving the problem for $V_1(x)$ and x_q^* using the dynamic constraint (9) with the two free boundary conditions (6) and (7). With the chosen form of $F_1(x(t))$ and $S(x(t))$ we obtain the following Bellman equation

$$rV_1 = \mu x V_1' + \frac{1}{2}\sigma^2 x^2 V_1'' + \left(1 - \exp(-\lambda x)\right) - \mu x \quad (23)$$

with the same boundary conditions as in the case of geometric mean reversion, (11) and (12). We solve this free boundary value problem numerically in a similar manner as before.

Once $V_1(x)$ is obtained, we proceed to find $V_0(x)$ and x_s^* under the dynamic constraint (8) and the two free boundary conditions (4) and (5). This problem leads to the following Bellman equation

$$rV_0 = \alpha x V_0' + \frac{1}{2}\sigma^2 x^2 V_0''. \quad (24)$$

Equation (24) is the same as in the classical real option problem with geometric Brownian motion describing the project value (see Dixit and Pindyck (1994)) except that the free boundary conditions are changed.

The elementary solution of (24) is of the type

$$V_0(x) = A x^{\beta}.$$

By substitution in (24), we obtain

$$\alpha\beta A x^{\beta} + \frac{1}{2}\sigma^2 \beta(\beta - 1) A x^{\beta} - r A x^{\beta} = 0$$

and hence we can identify β as the solution of the following quadratic equation

$$\frac{1}{2}\sigma^2 \beta(\beta - 1) + \alpha\beta - r = 0.$$

The two solutions are

$$\beta_1 = \frac{-(\alpha - \frac{1}{2}\sigma^2) + \sqrt{(\alpha - \frac{1}{2}\sigma^2)^2 + 2\sigma^2 r}}{\sigma^2}, \quad (25)$$

$$\beta_2 = \frac{-(\alpha - \frac{1}{2}\sigma^2) - \sqrt{(\alpha - \frac{1}{2}\sigma^2)^2 + 2\sigma^2 r}}{\sigma^2}. \quad (26)$$

This leads to a general form

$$V_0(x) = A_1 x^{\beta_1} + A_2 x^{\beta_2}.$$

It is obvious that β_2 is negative and for the same reason as before, i.e. to ensure the finite value of the project at $x = 0$, we must have $A_2 = 0$. The solution of (24) is therefore

$$V_0(x) = Ax^{\beta}, \quad (27)$$

where $A = A_1$ is a constant that is yet to be determined and $\beta = \beta_1$.

The two free boundary conditions (4) and (5) are translated to

$$Ax_s^{*\beta} = V_1(x_s^*) - I_s, \quad (28)$$

and $$\beta A x_s^{*(\beta-1)} = \left(V_1(x_s^*)\right)'. \quad (29)$$

To solve for $V_0(x)$ and x_s^* we proceed in the same way as before. Note that in (29) $V_1'(x)$ is once more replaced by the corresponding finite difference.

3. Numerical results

In this section we discuss the results of our numerical computations which are based on the previous two sections. We also undertake a comparative analysis and discuss in particular how the threshold levels x_s^* and x_q^* are affected by changes in the parameters. In addition, we investigate the relationship between uncertainty and the expected time the private company will develop the project under the concessionary agreement.

3.1. Geometric mean reversion. In the following we discuss the geometric mean reversion based model (1)-(3) with parameters $\lambda = 0.4$, $r = 0.2$, $\delta = 0.05$, $\mu = 0.2$, $\sigma = 0.1$, $\alpha = 0.2$, and $I_s = 0.7$. The numerical results obtained for $V_0(x)$, x_s^* as well as $V_1(x)$, x_q^* are displayed in Figure 1 and Figure 2 in the appendix.

Figure 1 displays the value functions, $V_0(x)$ and $V_1(x)$, as a function of the level of development x. The dashed line in the figure represents the private company's value function before entering the concessionary agreement, i.e. $V_0(x)$, whereas the thick line represents the company's value function after investing in the project minus the sunk costs the company needs to pay when the company adopts the project, i.e. $V_1(x) - I_s$. Our computation shows that it is optimal for the company to invest in the project at the threshold of $x_s^* = 1.2182$.

Figure 2 displays the value function of the private company before ending the concessionary agreement, i.e. $V_1(x)$ (thick line) as well as the terminal payoff $x - I_q$ (dashed line), as functions of the level of development x. Our numerical results show that it is optimal for the private company to end the concessionary agreement once the level of development reaches the threshold, $x_q^* = 2.7104$.

3.1.1. Comparative analysis and effects on the thresholds. There are several parameters in the model that could affect the private company's value function, $V(x)$, and thus the thresholds, x_s^* and x_q^*. For each case, we investigate the changes in the thresholds and value functions, $V_0(x)$ and $V_1(x)$ with three different values for each parameter while the remainder of parameters are fixed. The results are displayed in Figures 3-10. The parameters that we consider are:

- uncertainty measured in terms of the instantaneous volatility σ,
- the private company contribution μ,
- the depreciation parameter δ, and
- the penalty fee I_q.

Figure 3 and Figure 4 show how σ affects the value functions and the thresholds whereas Figure 5 and Figure 6 demonstrate the impacts of μ on the results. Figure 7 and Figure 8 illustrate the change in the value functions and thresholds as a consequence of changes in δ. The penalty fee the private company needs to pay if it decides to end the concessionary agreement, i.e. I_q, also affects the results as shown in Figure 9 and Figure 10.

It can be seen in Figures 3 and 4 that the thresholds, x_s^* and x_q^*, are both increasing in σ. The larger σ is, the greater is the uncertainty component in the level of development. The effects of μ on the thresholds can be observed in Figures 5 and 6. Our results show that μ positively impacts x_s^* but negatively x_q^*. This is intuitive as μ represents the money the firm contributes to development, which positively affects the growth in development, but negatively the costs carried by the private company, part of which is recovered in the terminal payoff. The higher the costs, the less attractive is it to initially enter the concessionary agreement.

The impact of depreciation, δ, on the thresholds is displayed in Figures 7 and 8. The threshold to enter the concessionary agreement, x_s^*, is increasing in δ while the threshold for ending, x_q^*, is decreasing. As depreciation is essentially a cost that needs to be carried by the private company under the concessionary agreement, the intuition behind this result is essentially the same as in the previous paragraph.

Figures 9 and 10 display the effect of the penalty fee I_q the firm needs to pay if it would like to end the concessionary agreement. In Figure 9, we observe that the amount of the penalty does not have any significant effect on the threshold for entering but does affect the value function, $V_0(x)$. This is intuitive as the penalty fee is also a kind of cost that the private company has to carry. In Figure 10 we observe that the higher the penalty fee, the longer the firm developing under the concessionary agreement. This is because the firm needs to wait for higher terminal benefits $S(x_q^*)$ in order to compensate for the higher penalty fees.

Let us now consider the effect of the government contribution to development, α, as well as the sunk costs the company needs to pay once the company adopts the project, I_s. Figure 11 demonstrates that the threshold, x_s^*, is increasing in α. The larger α, the larger the drift term in the level of development x, which will increase x in average and, hence, positively affect the value function $V_1(x)$. The increase in x_s^* should not be interpreted that the private company becomes more cautious to entering the agreement, but simply that it expects the government to push up the level of development quickly to a higher level, at which it becomes more profitable for the private company to enter.

It is observed in Figure 12 that the critical value to adopt the project, x_s^*, has a positive relation to sunk costs, I_s. This can be understood as follows. The higher the sunk costs that the firm needs to pay once it enters the concessionary agreement, the less incentives it has to do so. The private company needs to wait longer to guarantee that the value of the project is high enough to cover the sunk costs once it adopts the project.

3.1.2. The effect of uncertainty on the expected length of the concessionary agreement. In the previous section, we examined the effects of each parameter on the thresholds, x_s^* and x_q^*. In this analysis we also included the uncertainty parameter σ. A related, but conceptually different question is to ask for how long on average will the concessionary agreement last. In this section we will determine the expected value of the difference τ_2 and τ_1 when optimal thresholds are applied under different levels of uncertainty. The complexity of our model prevents us to use analytic results about exit times, such as in Sarkar (2000), and instead we apply Monte Carlo simulation. As our problem is path-dependent, particular care has to be taken, on the problem of generating paths. Standard theory on numerical simulation of stochastic differential equations suggests the Euler-Milstein schemes, which features a strong convergence rate of 1. Using Euler-Milstein, the discrete approximations of $x(t)$ following (2) for $t < \tau_1$ and (3) for $t \geq \tau_1$ are

$$x(t+1) = x(t) + (\alpha - \delta x(t))x(t)\Delta t + \sigma x(t)\Delta W + \frac{1}{2}\sigma^2 x(t)[(\Delta W)^2 - \Delta t], x(0) = x_0, \quad (30)$$

and $$x(t+1) = x(t) + (\mu - \delta x(t))x(t)\Delta t + \sigma x(t)\Delta W + \frac{1}{2}\sigma^2 x(t)[(\Delta W)^2 - \Delta t], \text{ respectively.} \quad (31)$$

Given the two thresholds for entry and exit x_s^* and x_q^*, and an initial value of the project $x_0 < x_s^*$, we simulate the level of development of the project, $x(t)$, using (30) and (31) above for many times and take the average value of the time the concessionary agreement lasts, i.e. the time the paths have spent between x_s^* and x_q^*, as an approximate for the expected value $\mathsf{E}(\tau_2 - \tau_1)$.

For the purpose of illustration, the following parameter values have been chosen: $\lambda = 0.6$, $r = 0.2$, $\mu = 0.3$, $\delta = 0.1$, $\alpha = 0.3$, $I_q = 1$ and $I_s = 0.7$. In our simulation, we consider eleven different values of σ which are $0.1, 0.11, 0.12, \ldots, 0.2$. Figure 13 displays σ and the expected value $\mathsf{E}(\tau_2 - \tau_1)$. In the analysis in the previous section we have observed that the larger σ is, the higher x_s^* and x_q^*. Our results now show that $\mathsf{E}(\tau_2 - \tau_1)$ appears to be increasing in σ, meaning that concessionary agreements are likely to last longer, the higher the uncertainty is.

3.2. The case of geometric Brownian motion. We now consider the case (1), (8)-(9) where the level of development is modeled as a geometric Brownian

motion. In our numerical anlyisis we use the following parameter values: λ = 0.4, r = 0.2, μ = 0.17, σ = 0.1, α = 0.17, I_q = 1 and I_s = 0.7. and $I_s = 0.7$. The results are shown in Figure 14 and Figure 15, respectively.

Figure 14 displays the value functions, $V_0(x)$ and $V_1(x)$, as a function of the level of development x when x follows a geometric Brownian motion. In the figure the dashed line illustrates the company's value function before entering the agreement, $V_0(x)$, whereas the thick line illustrates the company's value function after entering the agreement minus the sunk costs the company has to pay when entering, $V_1(x) - I_s$. Our results shows that it is optimal for the company to enter the concessionary agreement if the threshold $x_s^* = 0.8422$ is reached.

Figure 15 displays the value function, $V_1(x)$ and terminal payoff, as a function of the level of development x. In the figure the thick line represents the value function of the company before ending the agreement, $V_1(x)$, while the dashed line represents the terminal payoff $x - I_q$. Our results show that it is optimal for the company to end the agreement if the threshold $x_q^* = 5.5876$ is reached.

3.2.1. Comparative analysis and effects on the thresholds. In this section, we vary the same parameters as in the geometric mean reversion case (except that there is no depreciation here) and study how the company's value function $V(x)$ and threshold levels x_s^* and x_q^* are affected.

Figures 16-21 show the value functions, $V_0(x)$, $V_1(x)$ and terminal payoff as well as thresholds, x_s^* and x_q^* depending on the various parameters.

Figures 16 and 17 show how σ affects the value functions and the thresholds whereas Figures 18 and 19 illustrate the changes in the value functions and the thresholds caused by changes in μ. The effect of the penalty fee that the private company needs to pay to the government if it decides to end the agreement, I_q, can be seen in Figures 20 and 21.

The effect of the government contribution α as well as the sunk costs the company needs to pay once the company enters the agreement, I_s, also affect the results but only prior to entering, i.e. $V_0(x)$ and the threshold, x_s^*. The effects of changes in α and I_s are displayed in figure 22 and figure 23.

Comparing to the geometric mean reversion case, we find that the parameters, σ, μ, I_q, α and I_s, have the same qualitative effects on the thresholds, x_s^* and x_q^*. The same intuition applies.

3.2.2. The effect of uncertainty on the expected length of the concessionary agreement. As in the geometric mean reversion case, we are interested in the impact of uncertainty, measured in terms of the instantaneous volatility σ, on the time $\tau_2 - \tau_1$ the concessionary agreement is intact. As before the idea is to find the expected value of $\tau_2 - \tau_1$ using Monte Carlo simulation, while employing the Euler-Milstein method. In this case $x(t)$ follows (8) for $t < \tau_1$ and (9) for $t \geq \tau_1$. The Euler-Milstein scheme is given

$$\text{as } x(t+1) = x(t) + \alpha x(t)\Delta t + \sigma x(t)\Delta W + \frac{1}{2}\sigma^2 x(t)[(\Delta W)^2 - \Delta t], x(0) = x_0 \quad (32)$$

$$\text{and } x(t+1) = x(t) + \mu x(t)\Delta t + \sigma x(t)\Delta W + \frac{1}{2}\sigma^2 x(t)[(\Delta W)^2 - \Delta t] \text{ respectively.} \quad (33)$$

Given the two thresholds for entry and exit, x_s^* and x_q^*, and an initial value for the level of development, $x_0 < x_s^*$, we use Monte-Carlo simulation as in the geometric mean reversion case in order to compute $\mathrm{E}(\tau_2 - \tau_1)$. In our numerical example we have chosen the following set of parameters: λ = 0.4, r = 0.2, μ = 0.18, α = 0.18, I_q = 1 and I_s = 0.7. We use eleven different values for σ which are $0.1, 0.11, 0.12, \ldots, 0.2$. Figure 24 displays σ and the expected value of $\tau_2 - \tau_1$. The result shows the same relationship between σ and $\mathrm{E}(\tau_2 - \tau_1)$ as in the geometric mean reversion case, i.e. the higher the uncertainty is, the longer the concessionary agreement is expected to last.

Conclusion

We have studied the problem where a private company is given a concession by the state government to develop a project. The optimal time to enter the concessionary agreement and the optimal time to end it have been computed using different modeling assumptions, i.e. geometric mean reversion and geometric Brownian motion. The effect of various model parameters on the threshold

levels for entry and exit has been analyzed with particular emphasis on the uncertainty parameter σ. Further, the effects of uncertainty on the expected time the concessionary agreement is expected to last have been investigated by means of Monte Carlo simulation.

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Appendix A. Graphical illustration

The case of geometric mean reversion

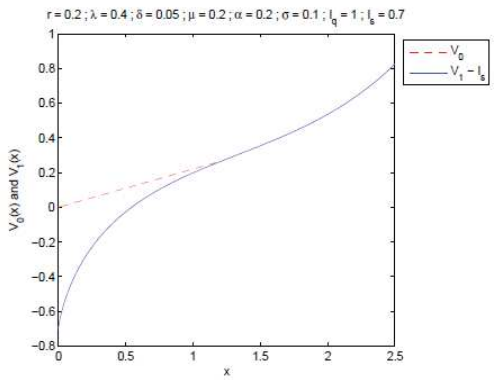

Fig. 1. GMR: Value functions; $V_0(x)$ and $V_1(x)$, and the threshold x_s^* for entering the concessionary agreement

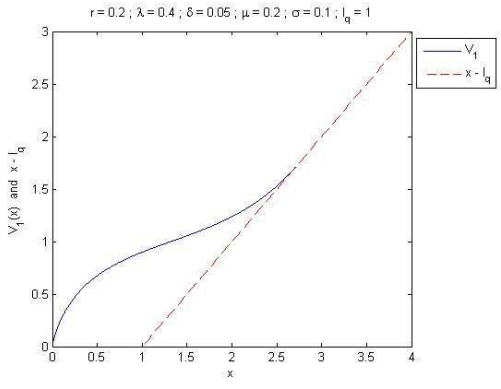

Fig. 2. GMR: Value functions; $V_1(x)$, and the terminal payoff, and the threshold x_q^* for entering the concessionary agreement

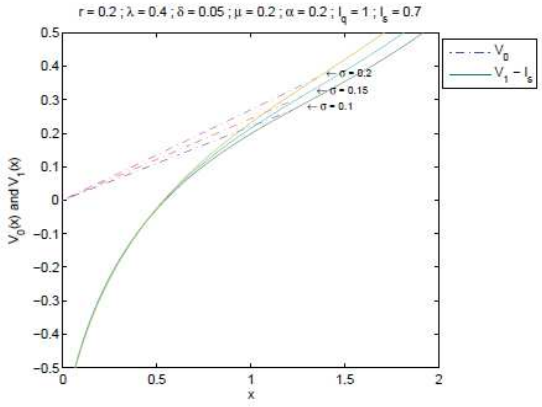

Fig. 3. GMR: Value functions; $V_0(x)$ and $V_1(x)$, and the threshold x_s^* for entering the concessionary agreement with different σ

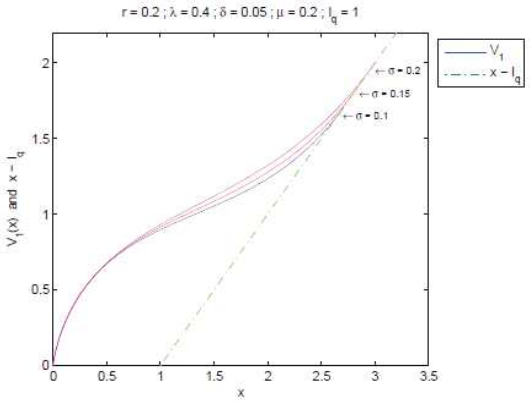

Fig. 4. GMR: Value function; $V_1(x)$, and the terminal payoff, and the threshold x_q^* for ending the concessionary agreement with different σ

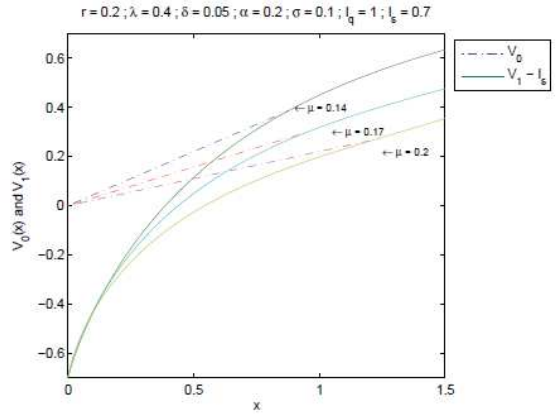

Fig. 5. GMR: Value functions; $V_0(x)$ and $V_1(x)$, and the threshold x_s^* for entering the concessionary agreement with different μ

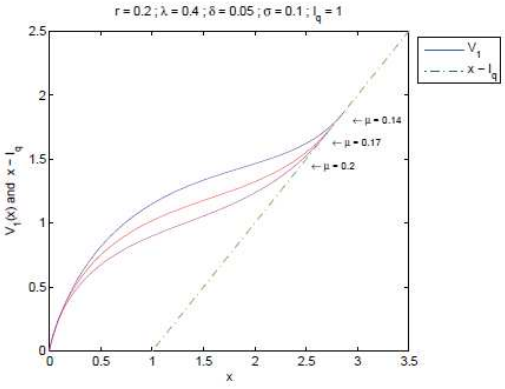

Fig. 6. GMR: Value function; $V_1(x)$, and the terminal payoff, and the threshold x_q^* for ending the concessionary agreement with different μ

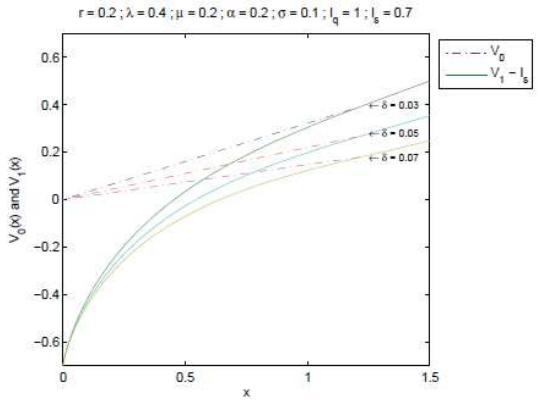

Fig. 7. GMR: Value functions; $V_0(x)$ and $V_1(x)$, and the threshold x_s^* for entering the concessionary agreement with different δ

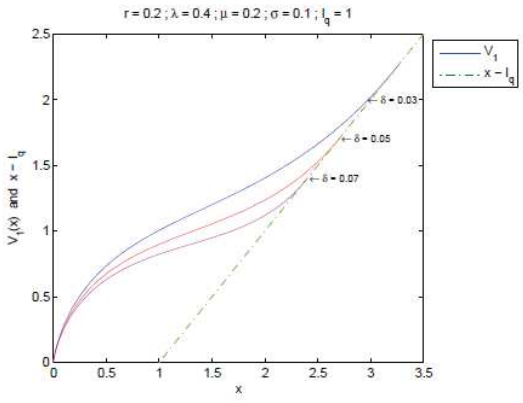

Fig. 8. GMR: Value function; $V_1(x)$, and the terminal payoff, and the threshold x_q^* for ending the concessionary agreement with different δ

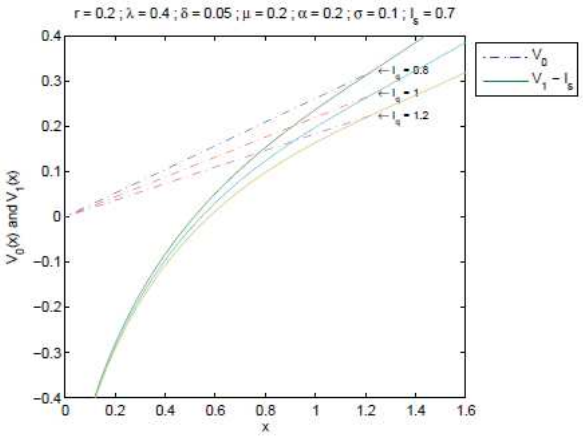

Fig. 9. GMR: Value functions; $V_0(x)$ and $V_1(x)$, and the threshold x_s^* for entering the concessionary agreement with different I_q

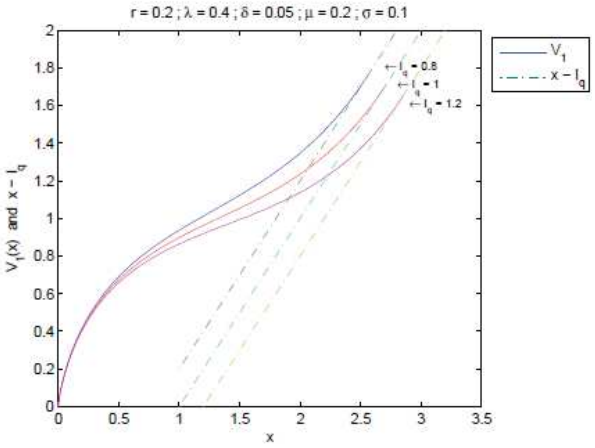

Fig. 10. GMR: Value function; $V_1(x)$, and the terminal payoff, and the threshold x_q^* for ending the concessionary agreement with different I_q

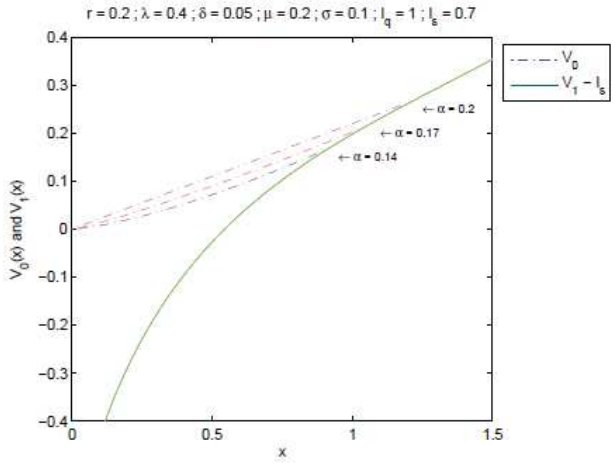

Fig. 11. GMR: Value functions; $V_0(x)$ and $V_1(x)$, and the threshold x_s^* for entering the concessionary agreement with different α

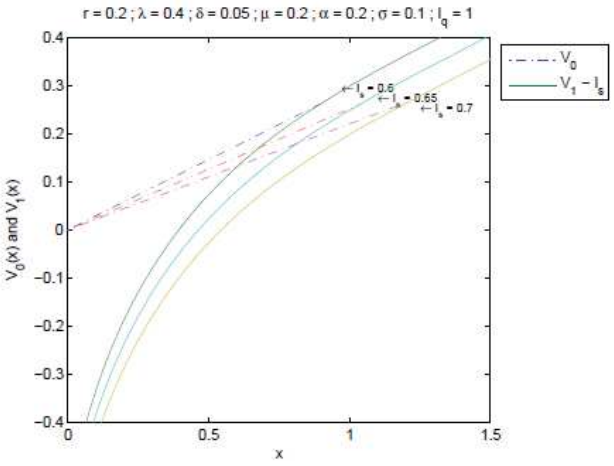

Fig. 12. GMR: Value functions; $V_0(x)$ and $V_1(x)$, and the threshold x_s^* for entering the concessionary agreement with different I_s

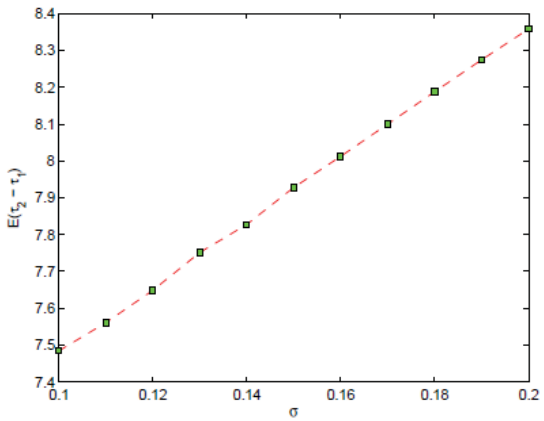

Fig. 13. GMR: Expected value of $\tau_2 - \tau_1$ with different σ

The case of geometric Brownian motion

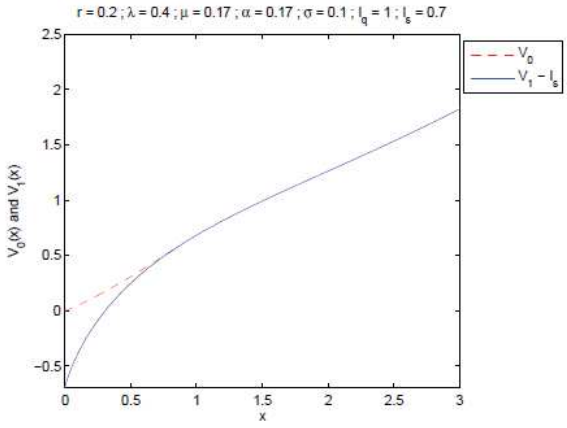

Fig. 14. GBM: Value functions; $V_0(x)$ and $V_1(x)$, and the threshold x_s^* for entering the concessionary agreement

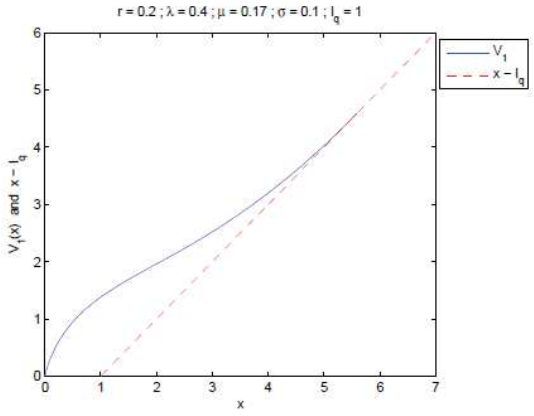

Fig. 15. GBM: Value function; $V_1(x)$, and the terminal payoff, and the threshold x_q^* for ending the concessionary agreement

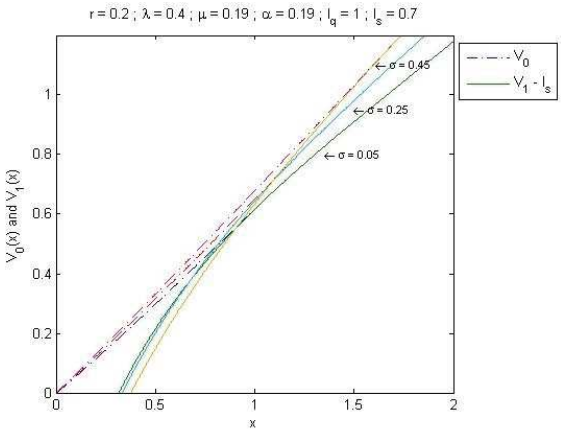

Fig. 16. GBM: Value functions; $V_0(x)$ and $V_1(x)$, and the threshold x_s^* for entering the concessionary agreement with different σ

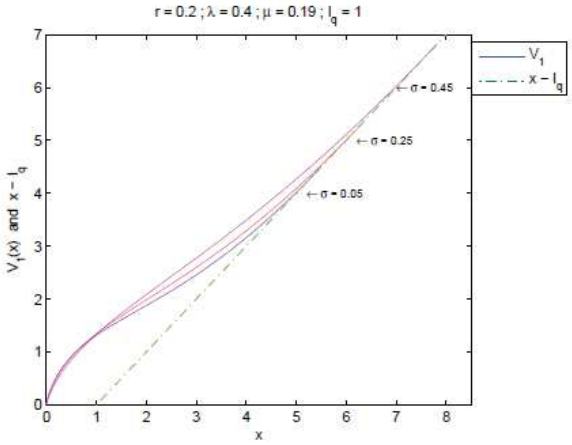

Fig. 17. GBM: Value function; $V_1(x)$, and the terminal payoff, and the threshold x_q^* for ending the concessionary agreement with different σ

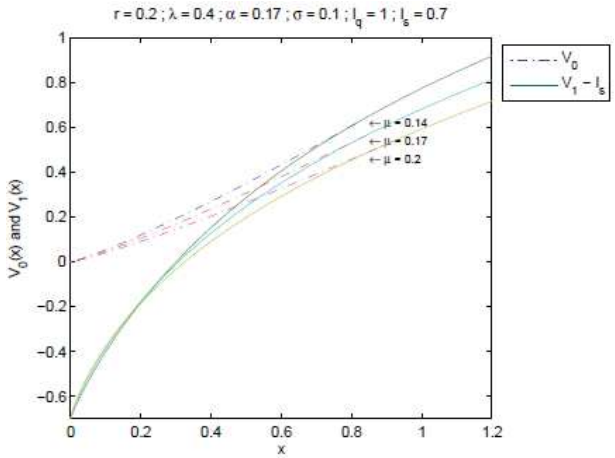

Fig. 18. GBM: Value functions; $V_0(x)$ and $V_1(x)$, and the threshold x_s^* for entering the concessionary agreement with different μ

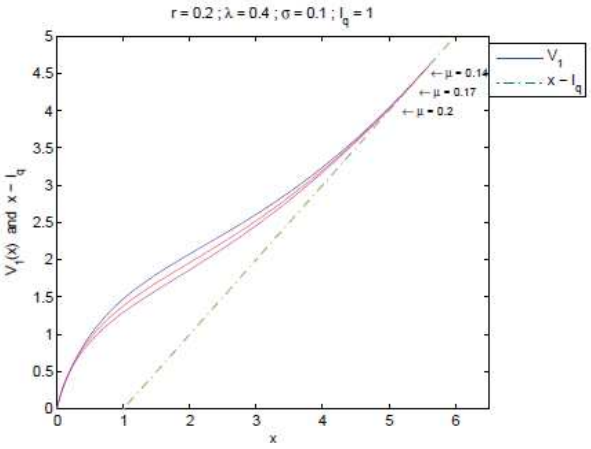

Fig. 19. GBM: Value function; $V_1(x)$, and the terminal payoff, and the threshold x_q^* for ending the concessionary agreement with different μ

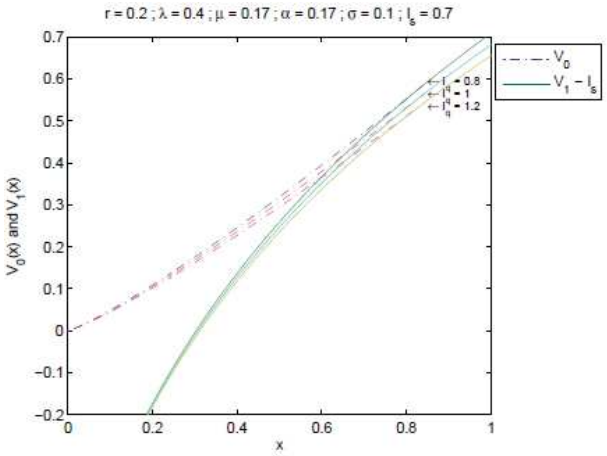

Fig. 20. GBM: Value functions; $V_0(x)$ and $V_1(x)$, and the threshold x_s^* for entering the concessionary agreement with different I_q

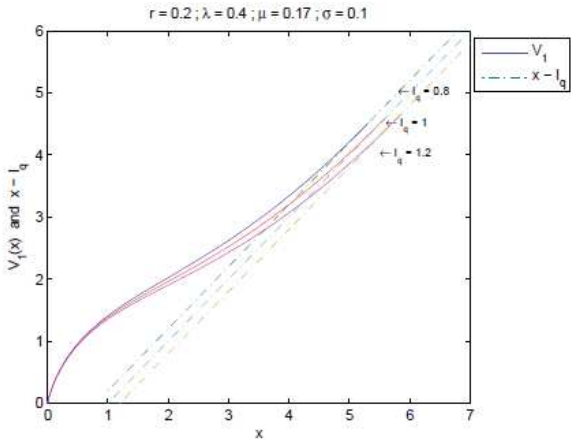

Fig. 21. GBM: Value function; $V_1(x)$, and the terminal payoff, and the threshold x_q^* for ending the concessionary agreement with different I_q

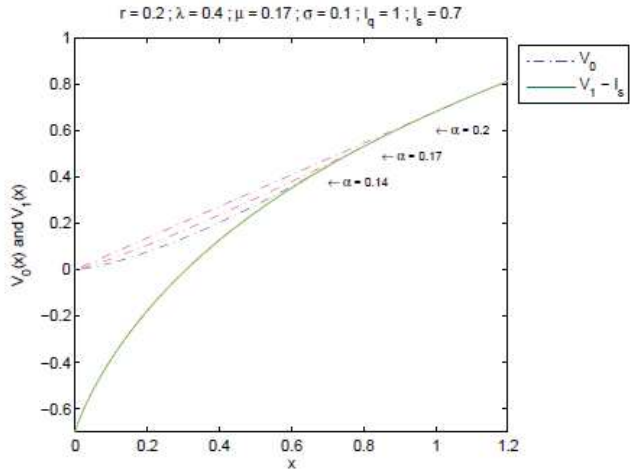

Fig. 22. GBM: Value functions; $V_0(x)$ and $V_1(x)$, and the threshold x_s^* for the concessionary agreement with different α

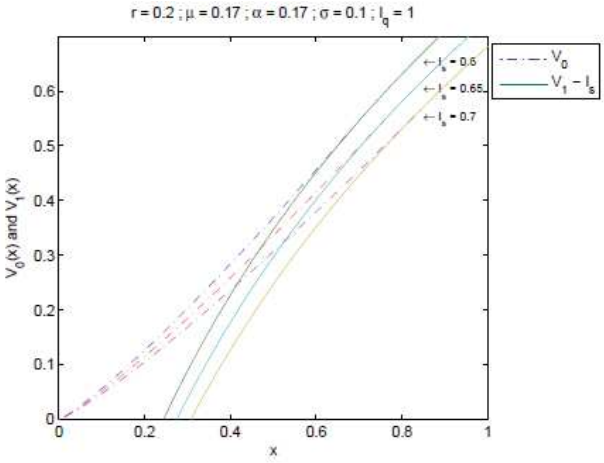

Fig. 23. GBM: Value functions; $V_0(x)$ and $V_1(x)$, and the threshold x_s^* for entering the concessionary agreement with different I_s

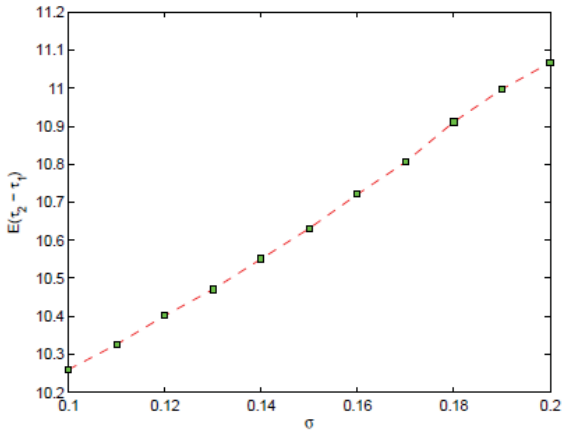

Fig. 24. GBM: Expected value of $\tau_2 - \tau_1$ with different σ

Appendix B. Shooting method

In this appendix, we summarize the idea of the so called "Shooting Method". Details can be found, for example, in Mathews and Fink (2004). The shooting method is a numerical method for solving a boundary value problem (BVP) by transforming it to an initial value problem (IVP) by making an initial guess for the first order condition. We then

solve the IVP by applying the time discretized numerical method like finite different method for example. Once the calculation is completed, we can verify whether it satisfies the desired boundary condition at the other endpoint or not. If not, the procedure is repeated after guessing a new first order condition. This proceeds until the boundary condition at the endpoint is ultimately satisfied up to an appropriate level of accuracy.

For the case of linear BVP, this procedure is simplified as closed-form solution of the BVP in terms of the solutions of the two corresponding IVPs can be obtained as shown in the following proposition. Note that the corresponding IVPs in general still have to be solved numerically.

Proposition let $x(t)$ denote the solution of the following linear BVP

$$x''(t) = p(t)x'(t) + q(t)x(t) + r(t) \quad ; x(a) = \alpha, x(b) = \beta. \tag{34}$$

Then $x(t)$ satisfies

$$x(t) = u(t) + \frac{\beta - u(b)}{v(b)} v(t), \tag{35}$$

where $u(t)$ and $v(t)$ are solutions of the IVPs (36) and (37) respectively:

$$u''(t) = p(t)u'(t) + q(t)u(t) + r(t); \; u(a) = \alpha, u'(a) = 0, \tag{36}$$

$$v''(t) = p(t)v'(t) + q(t)v(t); \; v(a) = 0, v'(a) = 1. \tag{37}$$

Proof: Since $u(t)$ and $v(t)$ are solutions of the IVPs (36) and (37) the linear combination

$$x(t) = u(t) + Cv(t) \tag{38}$$

is a solution of $x''(t) = p(t)x'(t) + q(t)x(t) + r(t)$ with boundary values

$$x(a) = u(a) + Cv(a) = \alpha + 0 = \alpha$$

$$x(b) = u(b) + Cv(b).$$

Choosing $C = \frac{\beta - u(b)}{v(b)}$ gives $x(b) = \beta$. Hence, the solution of (34) is given by

$$x(t) = u(t) + \frac{\beta - u(b)}{v(b)} v(t) \tag{39}$$

assuming that $v(b) \neq 0$.