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A stochastic model for loan interest rates

Abstract

The topic of interest rate restrictions and their legal implications represents a delicate subject about which a recent inventory of EU authorities was developed. This is aimed to inspection of the so called principle of "good morals" against usury for the Member States.

The most recent Italian law regulating legal rates of interest applied in loans, sets a threshold under which loan interest rates have to remain for being nonusurious, in the sense that if the loan rate lies outside the threshold, it becomes a usury rate and has to be prosecuted. The threshold is stated by Bank of Italy precisely each three-month period. In the paper the authors propose a stochastic process modelling the non-usurious interest rates applied in loans, in order to control its quarterly behavior. It is studied in the form of a modification of the Cox, Ingersoll and Ross model moving between two bands and closed expressions for its expected value and variance are given both conditional and unconditional. The model parameters are estimated by the Indirect Inference Method; the behavior of the expected value and variance functions are illustrated with graphs.

Keywords: Italian loan interest regulation, CIR model, conditional moments, indirect inference, Monte Carlo simulation.
JEL Classification: G17, G28.

Introduction

An inventory of interest rate restrictions against usury in the EU Member States was achieved at the end of 2010. In particular the EU authorities' attention focused on the Interest Rate Restrictions (IRR) established on precise legal rules restricting the credit price, both directly by fixed thresholds as well as indirectly by intervening on the calculation of compound interest (Directorate-General of the European Commission, 2011).

Since May 2011 the Italian law governs interest rates in loans with a new regulation, fixing a threshold above which interest rates applied in loans are considered usurious. The law fits in the civil and penal case in point of the usury crime the objective element of the interest rate over which penalties and civil sanctions go off. The borderline rate is calculated as 125 per cent of the reference rate TEGM plus 4 per cent. The acronym TEGM stands for Average Effective Global Rate. Therefore the maximum admissible value is:

Threshold rate = 1.25 *TEGM* + 0.04.

The law states that the difference between the threshold rate and TEGM cannot go further on 8%, this implying that the maximum value admissible for the reference rate cannot exceed 16%. TEGM value, quarterly settled and published by Bank of Italy, is based on annual interest rates applied by banks and financial intermediaries in financial operations of the same type, annually classified by Italian Ministry of Economics and Finance. TEGM includes commissions and general expenses while taxes are left out. In the framework the law prescribes, we propose a model for representing the stochastic movement of loan interest rates as a control tool for informing both in the loan interest rate trend and in option pricing forecasting.

In literature basic contributions to the study of the interest rates in a stochastic environment are well known and usually adopted in financial evaluation problems. In the particular case of exchange rates, the application to problems concerning the target zone had an important initial impulse with Krugman (Karatzas and Shreve, 1991) and from this contribute on, a large production has taken place, dedicated to both realignment and no realignment hypotheses. In particular De Jong et al. (Gouri´eroux and Monfort, 1996) presents a modification of the CIR model for exchange rates in a target zone in both cases.

Within the abovementioned Italian law, we propose a simply tractable stochastic model for controlling the behaviour of the loan interest rates in period of quarter; due to the nature of the process we are going to study, the model will be with no realignments, being no changes of the fixed TEGM in the each quarter time interval.

It is the aim of the paper to provide a stochastic representation of the evolution in time of non-usurious interest rates, in order to obtain information concerning returns/costs of loans.

The layout of the paper is the following. In section 1 we construct the model, set the stochastic differential equation of the process and show the existence and uniqueness of its solution. In section 2 we introduce a modification of the model serving as a means for simplifying moment calculation procedures. In section 3 we present stochastic calculus lines for getting conditional and unconditional expected value of the interest rate process while section 4 is dedicated to its variance. Section 5 is centred in the process parameter

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estimation and, finally, in section 6 numerical results are illustrated. Some concluding remarks close the paper.

1. The stochastic model for loan interest rates. The construction

We denote by i_t the stochastic interest rate applied in loans and by the TEGM, the constant quarterly rate fixed as previously explained. According to the present Italian regulation, i_t has to move within the following interval:

$$0 \le i_t \le 1.25\mu + 0.04 \quad (1)$$

in which the upper bound is known.

From equation (1) we can express the relation in term of the instantaneous interest rate:

$$0 \le \ln(1+i_t) \le \ln(1+\mu) + \ln\left(\frac{1+1.25\mu+0.04}{1+\mu}\right) \quad (2)$$

indicating by:

$$\delta_t = \ln(1+i_t)$$
$$m = \ln(1+\mu) \qquad m \in R^+ \quad (3)$$
$$k = \ln\left(\frac{1+1.25\mu+0.04}{1+\mu}\right) \qquad k \in R$$

equation (2) becomes:

$$0 \le \delta_t \le m+k \quad (4)$$

In what follows we describe the process δ_t as a stochastic interest rate constrained within the two bands in equation (4), both prefixed and known.

The description of this specific rate process asks for two basic behavioral conditions. The strong competitiveness among loan bidders in a sector characterized by an increasing transparency, makes suitable thinking to a process subject to the mean reverting elasticity property. In the model we propose, this aspect is represented by an autoregressive term attracting the interest stochastic rate i_t towards μ, that is δ_t towards m. Moreover we ask the model to be heteroskedastic: in particular, its diffusion coefficient has to decrease when approaching the threshold rate, that is when δ_t approaches $m+k$, and at the same time has to avoid negative values. No realignment problems are recognized: in each quarter time horizon, the model will not suffer changes in the fixed μ value, being TEGM constant in each period.

Basing on these considerations, the stochastic process for the loan instantaneous interest rate can be described by the following differential equation:

$$d\delta_t = -\rho(\delta_t - m)\,dt + \sigma\sqrt{\delta_t[k-(\delta_t - m)]}\,dW_t \quad (5)$$

with $0 \le \delta_t \le m + k$, m, p,σ and k positive parame ers, with δ_0 the initial position of the process and W_t a Wiener process.

The existence and uniqueness of the solution of equation 5 is based on well known results in stochastic differential equations theory.

2. The centred model

The model in equation (5) can be rewritten in the *centred version*. This handling allows the use of straightforward procedures useful for the moment calculations we will develop in sections 3 and 4.

Posing:

$$\delta_t^* = \delta_t - m \quad (6)$$

the stochastic differential in equation (5) becomes:

$$d\delta_t^* = -\rho\delta_t^* dt + \sigma\sqrt{(\delta_t^* + m)(k-\delta_t^*)}dW_t \quad (7)$$

with: $-m \le \delta_t^* \le k$.

The process δ_t^* is centred around m and the long term mean converges almost everywhere to 0. Moreover it is immediate to observe that the expected values of the processes in (5) and (7) only differ by m having nevertheless the same autocovariance function.

3. The expected value of the process

Applying Ito's theorem to equation (7), we have:

$$d\left(\delta_t^* e^{\rho t}\right) = \sigma e^{\rho t}\sqrt{(m+\delta_t^*)(k-\delta_t^*)}\,dW_t \quad (8)$$

and integrating in $(u,u+h]$[5] with $h \neq 0$ we can write:

$$\delta_{u+h}^* = \delta_u^* e^{-\rho h} + e^{-\rho h}\sigma \int_{(u,u+h]} e^{\rho(t-u)}\sqrt{(m+\delta_t^*)(k-\delta_t^*)}\,dW_t \quad (9)$$

Setting:

$$\varsigma_{u+h} = e^{-\rho h}\sigma \int_{(u,u+h]} e^{\rho(t-u)}\sqrt{(m+\delta_t^*)(k-\delta_t^*)}\,dW_t. \quad (10)$$

equation (9) can be expressed as follows:

$$\delta_{u+h}^* = \delta_u^* e^{-\rho h} + \varsigma_{u+h}. \quad (11)$$

Recalling a property of the martingale process (Gerber, 1979) and after some lines of algebra, we write the conditional expected value of the process:

$$E_u(\delta_{u+h}^*) = e^{-\rho h}\delta_u^* \quad (12)$$

from which we get immediately the unconditional expected values:

$$E(\delta_t^*) = 0,$$
$$E(\delta_m) = m. \quad (13)$$

4. The variance of the process

Let's consider the conditional variance of the process:

$$Var_u(\delta^*_{u+h}) = Var_u(e^{-ph}\delta^*_u + \varsigma^2_u + h) - E_u(\varsigma^2_{u+h}). \quad (14)$$

$$Var_u(\delta^*_{u+h}) = e^{-2ph}\sigma^2 \int_{(u,u+h]} [mk + (k-m)e^{-p(t-u)}\delta^*_u - e^{-2p(t-u)} - Var_u(\delta^*_u)]\,dt.$$

that, resorting to the properties of Ito stochastic integral, can be rewritten as (Gerber, 1979).

$$Var_u(\delta^*_{u+h}) = \int_{[u,u+h)e}^{2p(t-u)} E_u[(m+\delta^*_t)(k-\delta^*_t)]\,dt. \quad (15)$$

After some lines of algebra, we obtain:

Multiplying by e^{2ph} and setting:

$$g(h) = e^{2ph}\, Var_u(\delta^*_{u+h}) \quad (16)$$

we can write:

$$g'(h) = -\sigma^2 g(h) + \sigma^2[e^{2ph}mk + (k-m)\times \times e^{ph}\delta^*_u - \delta^{2*}_u] \quad (17)$$

with the boundary condition $g(0)=0$. It derives:

$$g(h) = e^{-\sigma h}[\sigma^2 mk \frac{e^h(\sigma^2+2p)-1}{\sigma^2+2p} + \sigma^2(k-m)\delta^*_u \frac{e^h(\sigma^2+p)}{\sigma^2+2p} - \delta^{2*}_u(e^{h\sigma^2}-1)]$$

from which, on the basis of equation (16), the conditional variance follows:

$$Var_u(\delta^*_{u+h}) = exp[-2ph]\, g(h) \quad (18)$$

and the unconditional variance directly flows:

$$Var_u(\delta^*_u) = \frac{\sigma^2 mk}{\sigma^2+2p}. \quad (19)$$

5. Parameter estimation

In models for which the likelihood function is analytically intractable or too difficult to evaluate, the indirect inference is a very useful simulation procedure. It was first introduced by (Smith 1990; 1993) and later extended in an interesting paper by (Gouriéroux, Monfort and Renault 1993).

The central idea is to use an auxiliary model which can be estimated using either the observed data or data simulated from the model itself. The aim is to find the parameter vector of the model so that these two sets of parameter estimates are as close as possible.

The process δ_t we are going to calibrate is observed at discrete times equally spaced and the approach we will follow consists in replacing the initial continuous model in equation (5) with its Euler discretization.

We recall that, referring to the general stochastic differential equation:

$$d\delta_t = \mu(t,\delta(t)) + \sigma(t,\delta(t))\,dW_t, \quad (20)$$

where W_t is a Wiener process, the Euler discretization can be expressed as follows:

$$r_t = r_{t-1} + \mu(r_{t-1};\theta) + \sigma(r_{t-1};\theta)\varepsilon_t, \quad (21)$$

where $\{r_t\}_{t=1,2,\ldots,T}$ are the available observations corresponding to the dates 1, 2, T, $\theta = [\rho, m, k, \sigma]$ is the parameter vector to be estimated and ε_k is a Gaussian white noise. Referring to model in equation (5), the Euler discretization can be written as follows:

$$r_t = r_{t-1} + p(m - r_{t-1}) + \sigma\sqrt{r_{t-1}(k-(r_{t-1}-m))\varepsilon_t}. \quad (22)$$

Indicating by Λ (r, θ) the likelihood function referred to the Euler discretization, we can estimate the model using the observed data. In particular we apply the maximum log-likelihood estimation method to the approximated model in equation (22) to get parameter estimates θ. Formally, θ' results:

$$\hat{\theta} = \arg\max_\theta \log(\Lambda(r,\theta)). \quad (23)$$

Referring to equation (21) we can rewrite equation (23) as follows (Ahangarani, 2005):

$$\theta = \arg\max_\theta \sum_{t-1}^{T}\{-0.5\log\sigma^2(r_{t-1};\theta) - \frac{1}{2}\times\frac{[r_{t-1}-\mu(r_{t-1};\theta)]^2}{\sigma^2(r_{t-1};\theta)}\}. \quad (24)$$

It is immediate to rewrite equation (24) considering the discretization introduced in equation (22).

Being the Euler discretization an approximation, the model in equation (22) is misspecified. By means of the indirect inference method, to correct the asymptotic bias of θ' we can use simulations performed under the initial model finding asymptotically consistent estimators of model in equation (5) (Ahangarani, 2005).

In order to simulate the continuous process we can use a finer Euler discretization involving a very small discretization step Δ such that:

$$r^{(\Delta)}_{(k+1)\Delta} = r^{(\Delta)}_{(k)\Delta} + \Delta\mu\ (r^{(\Delta)}_{(k)\Delta};\theta)\sqrt{\Delta\varepsilon^{\Delta}_{k}} \qquad (25)$$

where $t = k\ \Delta$ (k =1,2,…,T/Δ) and $\varepsilon^{\Delta}_{k} \approx N\ (0,\ \Delta)$, Given the set of random errors ε^{Δ}_{k} and the structural parameter vector θ, from equation (25) we generate M statistically independent simulated data sets, $\{\overline{r}^{(\Delta)m}_{k_\Delta}\} m = 1, 2, ..., M$, where $\{\overline{r}^{(\Delta)m}_{k\Delta}\} = (\overline{r}^{(\Delta)m}_{k\Delta}, \overline{r}^{(\Delta)m}_{2\Delta}, ..., \overline{r}^{(\Delta)m}_{T})$. Each of the M simulated data sets is built using the same set of random errors. At this point we will maximize the log of the likelihood function across the M simulations, getting the following equation:

$$\theta^{M} = \arg\max_{\theta} \sum\nolimits_{m-1}^{M} \sum\nolimits_{m-1}^{M} \{-0.5 \log \sigma^2\ (r^{m}_{t-1}(\overline{\theta});(\theta) - \frac{1}{2}\frac{[r^{m}_{t-1}(\overline{\theta}) - \mu(r^{m}_{t-1};\theta)]^2}{\sigma^2\ (r^{m}_{t-1}(\overline{\theta});\theta)}\}. \qquad (26)$$

The last step is the calibration of parameter estimates. We need to choose a formal metric to measure the 'distance' between θ' and θ'^{M} and to this aim we implement the Wald approach consisting in choosing in equation 26 such that the quadratic form in the vector (θ' – (θ'^{M}($\overline{\theta}$)results minimized (Keane and Smith, Jr., 2004):

$$\hat{\theta}^{Wald} = \arg\min_{\hat{\theta}} (\theta' - (\theta'^{M}(\overline{\theta}) W (\theta' - (\theta'^{M}(\overline{\theta})) \qquad (27)$$

having indicated by W a positive definite 'weighting' matrix. It has been shown that for T sufficiently large, the choice of W can be arbitrary (Gourieroux and Monfort, 1996). So we can assume W an Identity matrix.

5. Some results

In this section we illustrate the behaviour of the conditional expected value and variance functions.

The parameter estimation procedure described in section 4 has been applied to the historical series of the Italian average rates on loans referring to the period 1/04/1997-1/07/2011. Being the optimization procedure very time consuming we choose to simulate M = 50 paths. The optimization routine was implemented on *Matlab*. The initial values for the estimation procedure have been obtained by the historical series using the Least Squares methods implemented on *E-Views* 5.0. The procedure gives the following results for the parameters of the model in equation (5): p = 0.1165, with p =0.1165 m = 0.0575, k= 0.050371, σ = 0.0035.

The behavior of the conditional expected value of the centred rate δ^{*}_{t} is shown considering daily observations within the quarter and different values of the rate observed at the beginning of the period. In Figure1 we illustrate the trend of the process δ^{*}_{t} as function of the time and of the initial state values.

Fixing the time of valuation, the conditional expected value increases with the rate observed at the beginning of the period. On the other hand, fixing the initial rate, when the time increases the expected value decreases and tends to zero, the unconditional expected value. For low initial rates, expected δ^{*}_{t} rapidly increases at the beginning of the period and we can also observe that it is noticeably different from 0 as far as the initial part of the quarter for any initial rate value.

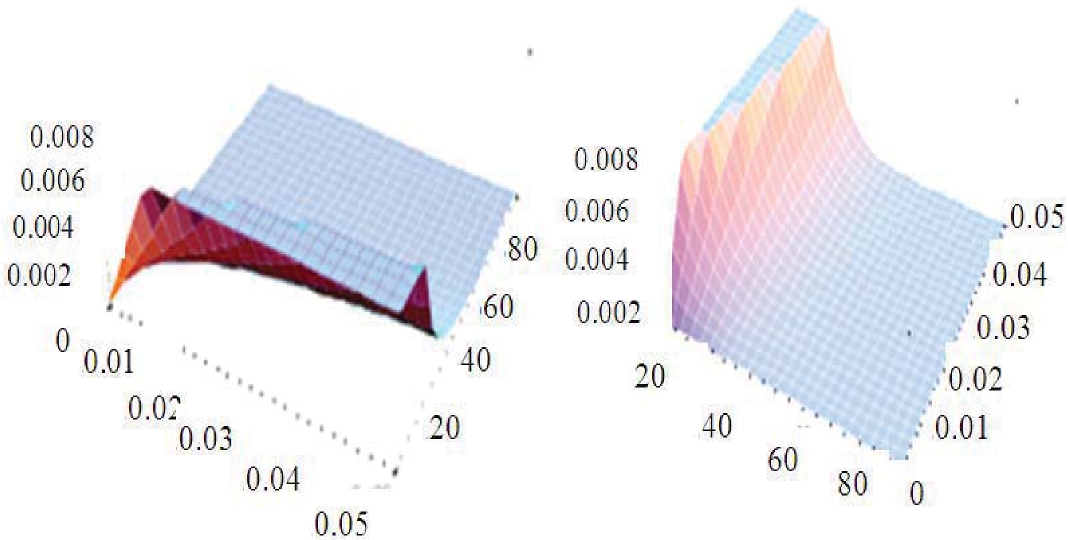


Fig. 1. Expected value function {δ^{*}_{u} 0,0. 05} {*h*, 1,90 days}

Figure 2 below shows the behavior of the variance function. Fixing the rates, the variance increases with time while it slightly decreases as function of the initial rate. The variance values tend to stabilize and converge to the unconditional variance value.

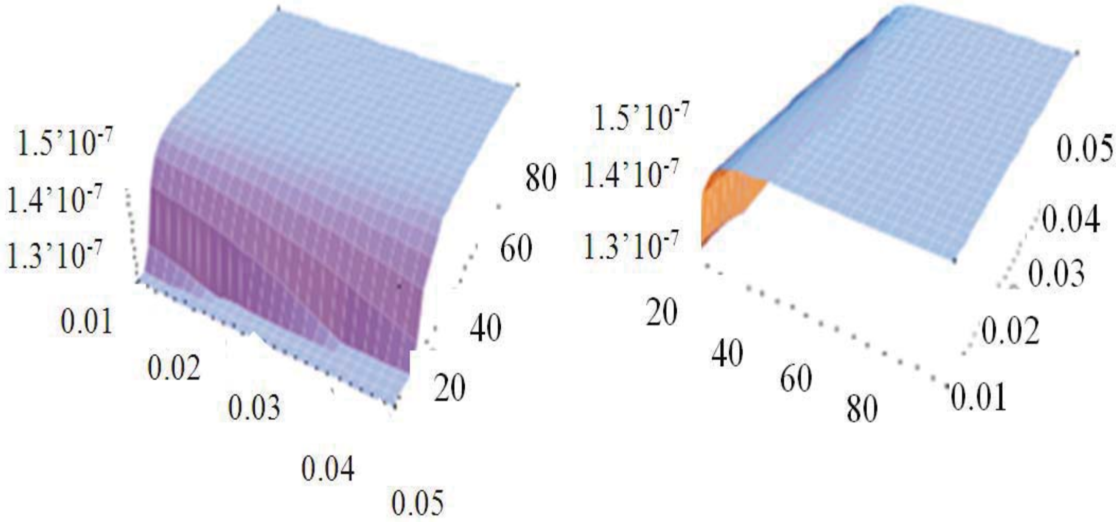
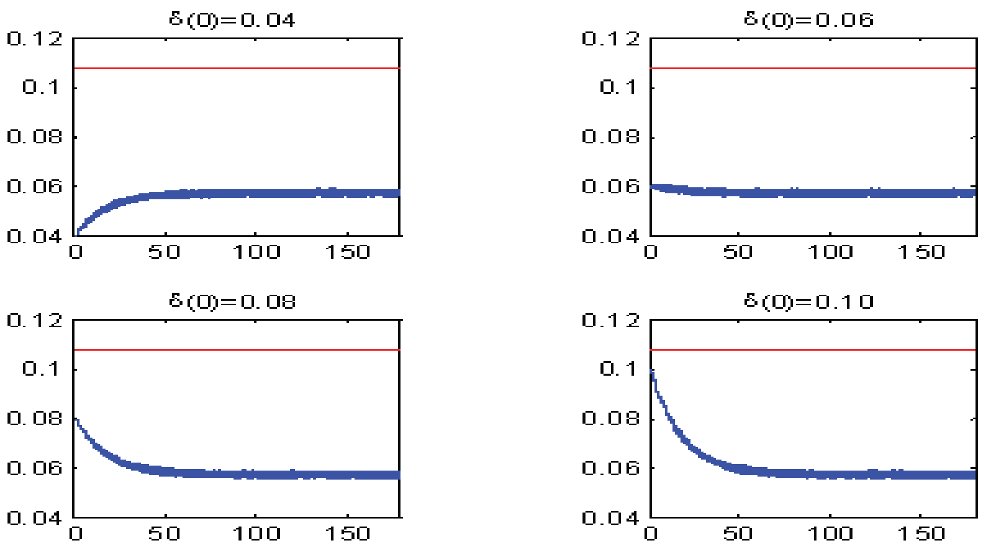


Fig. 2. Variance function{δ^*_u,0,0. 05} {*h*, 1,90 days}

As already observed for the expected value, the variance too is subjected to considerable variations during the initial part of the quarter.

In Figure 3 we refer to the stochastic process δ_t described by (5). We fit the model considering the last quarter of the data set used to estimate parameters. The red line is the actual maximum cap. We exemplify the trends of the process δ_t choosing four different values of the initial rate and simulate 10000 trajectories of δ_t, employing the euler discretization (22). We observe that, as the time increases, all the values tend to the mean m = 0.0575, that is to the expected value of the process. Moreover for each time of valuation, the process takes values lower than the cap and after 20 days the process tends to level off.



Note: Initial values: 4%, 6%, 8%, 10%.

Fig. 3. Simulated trajectories for δ (*t*)

Conclusions

The study provides micro-financial indications governing the relationship between creditor and debtor, which is represented by the interest rate with boundaries originated with legal measures.

This is useful for planning of financing transactions, within an environment constrained by benchmarks set by the law, even in the perspective of transactions within leveraged finance. The characteristic parameters, taken with respect to the time of evaluation, provide straight and synthetic addresses concerning the evolution in time of non-usurious interest rates, from the point of view of both parties.

The paper concerns the stochastic analysis of the behaviour of the loan interest rates according to the current Italian law, in the framework of the principles of fairness against usury, as it is consolidated in the EU member states. The study is restricted in a quarter time interval, coming into line with the time lag chosen by the legislator. In each three month period the upper threshold rate is fixed and consequently the loan rates, laying out of the this barrier, become usurious rates. In a three month period perspective, the system moves in a fixed target zone and, for its connection with the exchange rates modelling, is ranked in the no realignment cases.

The process describing the three month period behaviour of the loan rate is proposed as a modifica tion of the Cox Ingersoll and Ross model. It results as the unique solution of a stochastic differential equation and the expected value and the variance are calculated in conditional and unconditional hypothesis. The procedure for estimating the parameters characterizing the model is deepened and the numerical application closing the paper shows the results obtained for the parameters on the basis of an updated dataset of interest rates; the trends of expected value and variance of the process are shown with illustrations.

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