"A stochastic quantile approach for longevity risk"

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ARTICLE INFO	Albina Orlando, Giovanna Di Lorenzo and Massimiliano Politano (2014). A stochastic quantile approach for longevity risk. *Investment Management and Financial Innovations*, *11*(4-1)
RELEASED ON	Monday, 15 December 2014
JOURNAL	"Investment Management and Financial Innovations"
FOUNDER	LLC "Consulting Publishing Company "Business Perspectives"

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Abstract

This paper investigates the problem of quantifying longevity risk in a quantile perspective. In this field, the idea of deepening the expected changes of future mortality rates over a single year is gaining. In the following the authors propose an approach which combines a stochastic model for the evolution of mortality rates and a quantile analysis of the mortality distribution in order to capture the trend component of longevity. An ex post analysis is proposed, relying on the past mortality experience of the Italian male population measured in the period of 1954-2008. Numerical applications illustrate the results and their impact both on the survival probabilities and on the risk margin for the insurance company.

Introduction

During the 20th century, human life expectancy have considerably increased for the populations of many developed countries. Although the past trends suggest that further changes in the level of mortality are to be expected, the future improvements of life expectancy are uncertain and difficult to be predicted.

This uncertainty about the future development of mortality give rise to longevity risk. The real challenge for public pension systems and for private insurance companies consists precisely in the design of products able to absorb any adverse events concerning the future mortality. In other words, the challenge is how to deal with the longevity risk. When we treat benefits depending on the survival of a certain number of individuals, the calculation of the present values, used both for pricing and for reserving, requires an appropriate projection of mortality in order to avoid an underestimation of future costs. Therefore, actuaries have to employ projected life tables incorporating a forecast of future trends of mortality. The insurer bears the risk that the projections of mortality turn out to be incorrect and the annuitants live longer than expected. Different approaches for the construction of the projected tables have been developed until now, (for a full report on this subject, see Pitacco 2004), but no one turned out to be suitable for the problem solution. The problem is twofold. On the one hand, insurers have to make the annuities market attractive to the insured. At present, the risk borne out by insurers for insurance annuities, which is undoubtedly too high, is reflected in high premiums charged for these products that discourage individuals who are intending to purchase annuities. On the other hand, Solvency II regulation requires the constitution of appropriate margins that are difficult to bear for an insurance company.

For this reason, many insurance companies and pension funds providers focus in the issue of sharing the longevity risk. An ordinary way to solve this problem is through reinsurance, but this method often involves high costs. The securitization provides a viable alternative (see Denuit, Devolder and Goderniaux, 2007), but unfortunately the longevity bonds are not a very attractive business for investors. Denuit et al. in 2011 thought the reduction of annuity periodic payments in a similar way to what happens in the context of securitization. In this way the risk is shared between insurer and insured, but nevertheless we obtain a significant reduction of benefits for the insured. Richards et al. in 2014 proposed a very interesting idea based on the quantification of expectation of change in mortality over a one year horizon. Such an approach lies at the heart of the one year value at risk view of the actuarial liability. We try to develop this concept relying on past mortality experience of the Italian population measured in the period of 1954-2008. A computational tractable approach based on a CIR type stochastic process for modeling the future uncertainty about the force of mortality is used. Essentially we try to combine a stochastic model for mortality rates approach with a quantile simulation procedure for the short period survival probabilities in order to quantify the risk of the insurance position. This approach, which combines a stochastic model for the evolution of mortality rates and a quantile analysis for the mortality distribution, can be useful to capture the trend component of longevity. This also can help to minimize the security loading in order to front the insurer actuarial liability.

The paper is organized as follows. In section 1 the general issue of modeling the uncertainty in future mortality is fronted and a CIR type model for describing the future evolution of hazard rates is described. In section 2 the quantile analysis and the stochastic simulation procedure is introduced. In section 3 the authors look for the conditions that allow to quantify the longevity risk via quantile analysis. The final section concludes and discusses the results.

1. The mortality model

Let us consider an individual aged x in the calendar year t. $p_x(t) = P(T_x(t) > 1)$ is the probability that an individual reaches the age $x +1$. Analogously $_k p_x(t)$ is the probability that an individual aged x in year t reaches age $x + k$ in the year $t + k$. If we consider the hazard rate for an individual aged $x + t$ in the year t μ_{x+t} we have

$$_k P_x(t) = E\left[e^{-\int_0^k \mu_{x+s} ds} \right]. \qquad (1)$$

We describe the evolution in time of mortality by a widely used stochastic mortality model (see Biffis, 2005; Dahl and Moller, 2006), supposing that the force of mortality at time t for an individual aged $x+t$ is given by:

$$d\mu_{x+t} = \kappa(\gamma - \mu_{x+t}) dt + \sigma \sqrt{\mu_{x+t}} dB_t, \qquad (2)$$

where κ and σ are positive constants, γ is the long term mean and B_t is a standard Brownian motion. This model, referred as the CIR mortality model has the property that the mortality rates are continuous and remain positive. Moreover, for $2\kappa\gamma \geq \sigma^2$ the mortality rates does not reach zero, and the drift factor $\kappa(\gamma - \mu_{x+t})$ ensures the mean reversion of μ_{x+t} towards the long-term mean γ.

For convenience, we now introduce the centered version of the model. Let us consider the shifted $\mu_{x+t}^* = \mu_{x+t} - \gamma$. The process is then centred around γ and the long term mean converges almost everywhere to zero:

$$d\mu_{x+t}^* = \kappa\mu_{x+t}^* dt + \sigma \sqrt{\mu_{x+t}^* + \gamma} dB_t, \qquad (3)$$

with initial condition given by the known value of μ_{x+t}. Its solution is given by:

$$\mu_{x+t}^* = e^{-\kappa t}\mu_{x+0}^* + \sigma e^{-\kappa t}\int_0^t e^{\kappa u}\sqrt{\mu_{x+u}^* + \gamma} dB_u. \qquad (4)$$

The expected value, the covariance and the stationary variance functions immediately follow:

$$E[\mu_{x+t}^*] = e^{-\kappa t}\mu_{x+0}^*$$

$$cov(\mu_{x+t}^*, \mu_{x+s}^*) = \sigma^2 \frac{e^{-\kappa t} - e^{-\kappa(s+t)}}{\kappa}\mu_{x+0}^* +$$

$$+\sigma^2 \frac{e^{-\kappa(t-s)} - e^{-\kappa(s+t)}}{2\kappa}\gamma, \quad s \leq t,$$

$$\lim_{t->\infty} Var[\mu_{x+t}^*] = \frac{\gamma\sigma^2}{2\kappa}.$$

1.1. Parameter estimation procedure. Estimating the parameters of the stochastic mortality model requires the discrete representation of the model.

To this aim, we refer to the covariance equivalence principle which requires that the expected values and the stationary variances of the continuous and discrete processes to be equal.

The discrete model representation is given by the following equation:

$$\mu_{x+t}^* = \varphi\mu_{x+t-1}^* + \sigma_a \sqrt{\frac{2\varphi}{1+\varphi}\mu_{x+t-1}^* + \gamma} a_t. \qquad (5)$$

The expected value, the covariance and stationary variance functions of the previous equation are:

$$E[\mu_{x+t}^*] = \varphi^t \mu_{x+0}^*,$$

$$cov(\mu_{x+t}^*, \mu_{x+s}^*) = 2\varphi^t \sigma_a^2 \mu_{x+0}^* \frac{1-\varphi^s}{1-\varphi^2} +$$

$$\varphi^{t-s}\sigma_a^2 \gamma \frac{1-\varphi^{2s}}{1-\varphi^2}, \quad s \leq t,$$

$$\lim_{t->\infty} Var[\mu_{x+t}^*] = \frac{\sigma_a^2 \gamma}{1-\varphi^2}.$$

The estimation procedure starts by finding the value of φ that minimizes the residual sum of squares function:

$$RSS = \sum_{t=1}^{N} \frac{(\mu_{x+t}^* - \varphi\mu_{x+t-1}^*)^2}{\frac{2\varphi}{1+\varphi}\mu_{x+t}^* + \gamma}.$$

The least squares estimate of σ_a^2 is given by RSS/N-1.

Finally the continuous model parameters are obtained by means of the parametric relationships between continuous and discrete models, derived by applying the covariance equivalence principle:

$$\begin{aligned} \varphi &= e^{-\kappa} \\ \sigma_a^2 &= \sigma^2 \frac{1-e^{-2\kappa}}{2\kappa}. \end{aligned} \qquad (6)$$

At this point, by the Pitman and Yor formula, we can compute:

$$_k P_x(t) = E\left[e^{-\int_0^k \mu_{x+s} ds} \right] = \frac{exp\left\{ -\frac{x}{\sigma^2} w \frac{1 + \frac{\kappa}{w}coth(w\kappa/2)}{coth(wk/2) + \frac{\kappa}{w}} \right\}}{\left(cosh(wt/2) + (\kappa/w)sinh\, coth(wk/2)\right)^{\frac{2\kappa\gamma}{\sigma^2}}}, \qquad (7)$$

where $x = \mu_0\, e$ $w = \sqrt{\kappa^2 + 2\sigma^2}$.

Applying the described estimation procedure, the significant parameters of the mortality-CIR model are obtained and therefore the survival probabilities for each specific calendar year.

Our set of data relates to the Italian male population with annual age-specific death counts ranging from ages 64 to 89 over the period of 1954 to 2008 (data source: Human mortality database www.morta-lity.org).

We refer to the class of the forward mortality models. These models study changes in the mortality rate curve for a specific age cohorts and capture dynamics of each age cohort over time for all ages greater than x in a specific year t (for example age x in the year t, $x + 1$ in the year $t + 1$ and so on). In this case, the mortality curves are modeled diagonally (for example see Dahl, 2004; Cairns et al., 2006; Bauer et al., 2008). In practice, on the basis of data available for the previous 25 years, we can estimate the model parameters for the year t and, as a result, it is possible to get the forecasted survival probabilities.

For example, with the data of the period of 1954-1978 it is possible to obtain the column of the survival probabilities for the year 1979. This procedure is repeated thirty times in order to obtain the annual survival probabilities over the period of 1979 to 2008 and ranging from ages of 64 to 89.

These probabilities can be compared with the corresponding survival rates obtained from the tables of the Human Mortality Database.

Regarding the choice of fixing the extreme age to 89, recent studies (Khalaf-Allah et al., 2006) have shown that the most damaging effects in terms of annuities present values for the provider are in the age range of 73-80. Clearly this happens because the number of survival is still large at these ages. As a consequence, even modest improvements in the level of survival probabilities with respect to those used for pricing and reserving, result in large additional costs for the annuity provider. The results of the estimation procedure are summarized in the following table (Tab. 1). The parameters κ and σ^2 are obtained, for each year, by means of the relations (5), after the estimation of the discrete parameters in (4). We choose to

calculate the long term mean γ as the simple mean of each historical series used to estimate the parameters.

κ takes the same value for each calendar year. The reason can be found in the high autoregressive parameter of the discrete model $\varphi = 0.999$, which is the same each year explaining the high correlation of each data of each series with the preceding one.

Table 1. CIR-estimated mortality parameters. Data source: Human Mortality Database: Italian male population

Year	κ	σ^2	γ
1979	0.0010005	0.02154137	0.09879589
1980	0.0010005	0.02218555	0.09870146
1981	0.0010005	0.02196340	0.09855553
1982	0.0010005	0.02125207	0.09849015
1983	0.0010005	0.02006831	0.09848095
1984	0.0010005	0.02260864	0.09951799
1985	0.0010005	0.02051935	0.09773577
1986	0.0010005	0.02120267	0.09732236
1987	0.0010005	0.01981722	0.09758413
1988	0.0010005	0.01874663	0.09654774
1989	0.0010005	0.01883434	0.09567750
1990	0.0010005	0.01846146	0.09354197
1991	0.0010005	0.01880755	0.09122329
1992	0.0010005	0.01876966	0.09087511
1993	0.0010005	0.01765508	0.09013384
1994	0.0010005	0.01824302	0.08918243
1995	0.0010005	0.01765857	0.08768303
1996	0.0010005	0.01786665	0.08616861
1997	0.0010005	0.01748889	0.08443631
1998	0.0010005	0.01756343	0.08338616
1999	0.0010005	0.0182456	0.08166418
2000	0.0010005	0.01765565	0.07982632
2001	0.0010005	0.0170472	0.07782871
2002	0.0010005	0.01693314	0.07627881
2003	0.0010005	0.01732366	0.07500763
2004	0.0010005	0.01858599	0.07417957
2005	0.0010005	0.01731115	0.07317319
2006	0.0010005	0.01812203	0.07046889
2007	0.0010005	0.01662066	0.06851038
2008	0.0010005	0.01709801	0.06566561

Figures 1, 2 and 3 show the comparison between the estimated annual survival probabilities obtained by means of the CIR model and the corresponding probabilities of the Italian male population. The results are shown year by year over the period of 1979-2008.
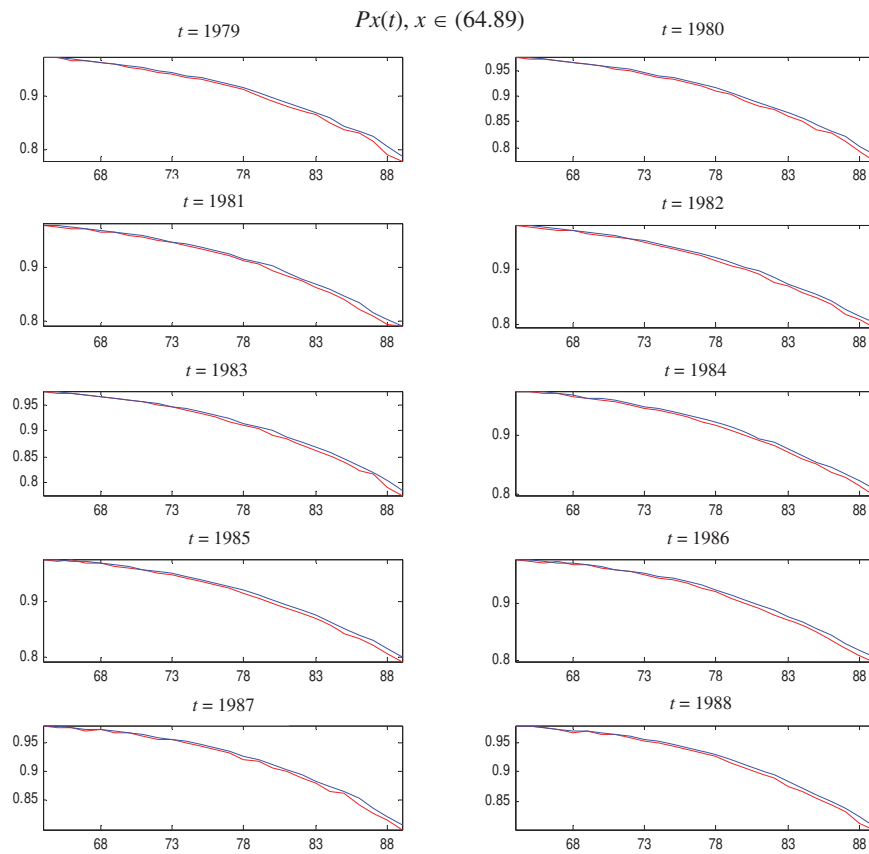
$Px(t)$, $x \in (64.89)$

Fig. 1. Annual survival probabilities $Px(t)$ with $x \in (64.89)$ for each calendar year t ranging from 1979 to 1988. Comparison between CIR model (upper line) and real data (lower line)
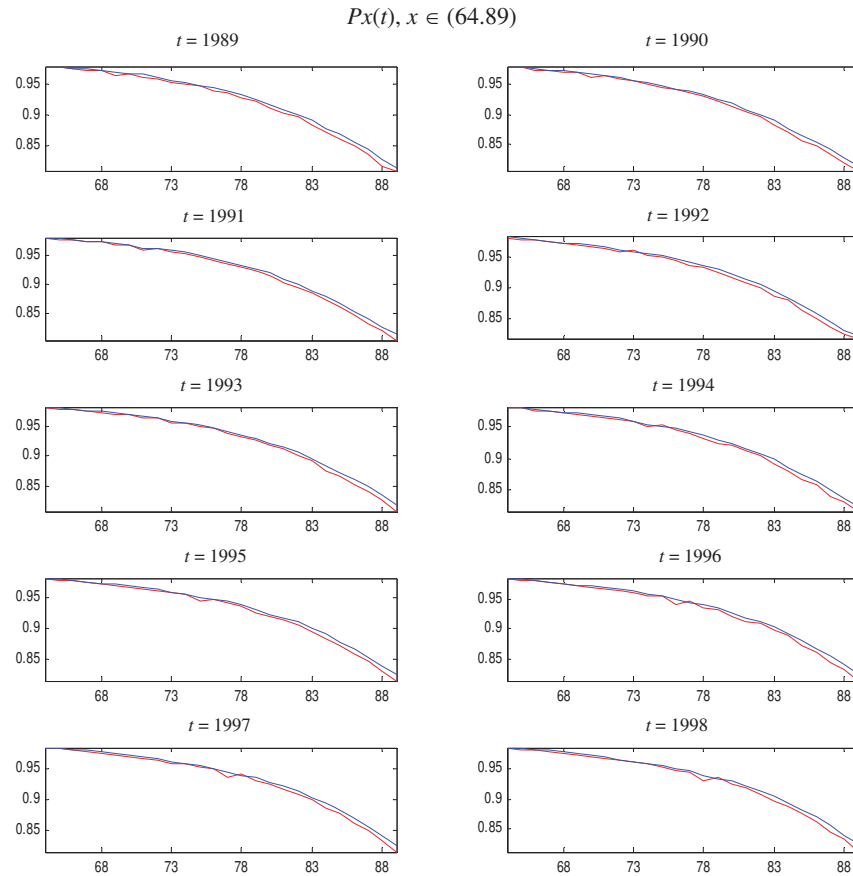
$Px(t)$, $x \in (64.89)$

Fig. 2 Annual survival probabilities $Px(t)$ with $x \in (64.89)$ for each calendar year t ranging from 1989 to 1998. Comparison between CIR model (upper line) and real data (lower line)
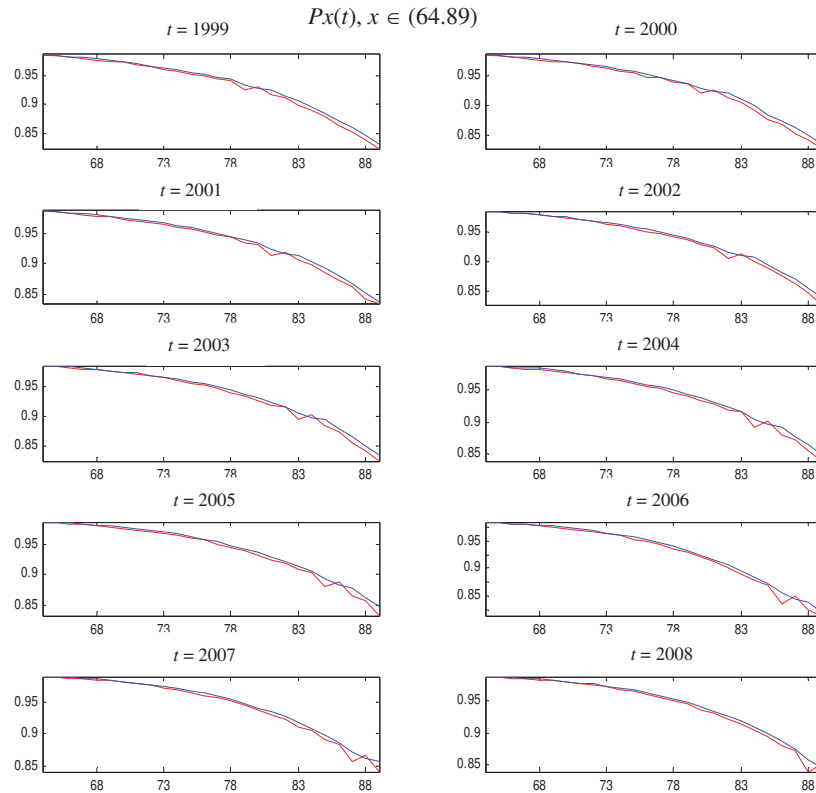
$$Px(t), x \in (64.89)$$

Fig. 3. Annual survival probabilities $Px(t)$ with $x \in (64.89)$ for each calendar year t ranging from 1999 to 2008. Comparison between CIR model (upper line) and real data (lower line)

Now, on the basis of mortality data for the last 25 years, the model is able to provide a good fit to the real survival probabilites of the next year but, unfortunately, fails in projection. In other words, it is not able to capture the decrease in time of the parameters κ and γ because of the well known phenomena of rectangularization and expansion of the Lexis point. For this reason, we try to combine a stochastic model for the evolution of mortality rates with quantile analysis for the mortality distribution in order to capture the trend component of longevity.

2. The quantile analysis

The quantile estimation gives an important information to the insurer by quantifying the tail events. In our analysis we refer to a tail event as the event of a survival probability higher than the expected one. This is crucial for the insurer. As well as the uncertain phenomena on the life expectancy, it is necessary to quantify the effects due to possible unexpected tail events. Only the awareness of the additional element can help to fully address the longevity risk.

We consider the survival probabilities $_t p_x$, derived by the stochastic model described in the previous section. Fixing x we derive, in a one year horizon, a set of cumulative probabilities and we estimate the related quantiles. To this end we resort to a stochastic simulation procedure and derive, for the following year a set of cumulative probabilities. In

particular, we simulate a large number N of sample paths, each of one producing a simulated set of $_t p^s_x$ $(S = 1,…, N)$.

The mortality risk measure we refer to is the quantile:

$MRM = q_\alpha,$

where α is the confidence level.

In practice, we consider the stochastic differential equation (2) and implement a Monte Carlo simulation procedure for the demographic quantities. In order to perform the simulation procedure it is necessary to consider the discrete time equation for the chosen stochastic differential equation describing the evolution in time of the mortality rates. On the basis of the first order Euler discretization of equation (7), with a time interval $[x + t, x + t + 1]$ we have:

$$\mu_{x+t_{k+1}} = \mu_{x+t_k} + \kappa(\mu - \mu_{x+t_k})\Delta + \sigma\sqrt{\mu_{x+t_k}\Delta} \cdot \varepsilon_k,$$

$$k = 1, 2,.., n\text{-}1, \tag{8}$$

where $\Delta = 1/n$ is the sampling interval, with ε_k being the increment ΔB_k of the Wiener process between $t_{k+1} = (k + 1)\Delta$ and $t_k = k\Delta$. The increments ΔB_k are $N(0, \Delta)$ distributed random variables. The discretized process is then represented by the sequence $\{\mu_{t_1}, \mu_{t_2}, ..., \mu_{t_n}\}$.

We know that the relation between the survival probabilities and the force of mortality is given by:

$$_k p_x(t) = E\left[e^{-\int_0^k \mu_{x+s} ds} \right], \tag{9}$$

with $_k p_x(t) = exp\left(-\sum_{k=1}^{n-1} \mu_{t_k} \cdot \Delta\right). \tag{10}$

The following simulation procedure is carried out:

1. generation of a sequence of n pseudo random numbers $\{\varepsilon_k\}_{k=1, 2, ..., nT}$, N(0, Δ) distributed;
2. computation of one simulated path for the stochastic mortality rates on the basis of equation (8);
3. computation of the discounting factors by means of equation (10);
4. computation of the survival probabilities using the estimated mortality table.

The simulation procedure is repeated N times to gain N values for the survival probabilities.

At this point, in a year by year valuation, we can estimate the survival probability quantiles at time t.

3. Longevity risk management via quantile analysis

The above analysis shows that the model fails to capture the trend of longevity. Although there is a good fit to the data in a year to year perspective, using the probability of the model and comparing them with those available on the tables ex post, you get ratios that are significantly less than one, in the sense that the probability provided by the model are lower than those observed in reality. At this point, we combine the model chosen for the evolution of mortality with a quantile analysis for the mortality distribution. Referring to equation (2), a large number of paths for the force of mortality are simulated. Each path allows to compute a simulated set of probabilities, that is $p_{x+j}(t + j)$ with $x = 64$, $t = 1979, 1980, 1981, 1982$, and $j = 0, 1, 2,…, 24$. For each simulated set, we study the ratio between the simulated survival probability and the "*true*" probability, that is the probability detected ex post by the life table. Clearly, it is desirable this ratio is equal to one, so that we study the confidence level that makes this value equal to one. This level will provide an estimate of the goodness of fit of the simulations but, above all, it will provide an estimate of the probability of underestimating the "*true*" probability.

As one can see, the response of the model is good in the medium term. For all the years considered the results of the reports are quite close to 1 up to age 78. The probability of underestimating the survival probabilities is negligible. The situation is different for age greater than 78. In this case the probability of underestimating grows up to 24%. Around this level the probability of underestimation stabilizes even for higher age. At this point it is possible to draw two conclusions. The first is that the combination between a stochastic model for the evolution of the force of mortality and a quantile approach allows to control the deviations of mortality from its expected trend due to the longevity and to limit the probability of underestimation within precise limits. The second is that the introduction of a single threshold to describe uncontrolled deviations of mortality from its trend looks wrong. The constitutions of the funds by the insurance company should be more responsive to its risk profile.

Conclusive remarks

In this paper we investigated the problem of quantifying longevity risk in a quantile perspective. In particular we proposed an approach which combines a stochastic model for the evolution of mortality rates and a quantile analysis for the mortality distribution in order to capture the trend component of longevity. We performed an ex post analysis relying on the past mortality experience of the Italian male population measured in the period of 1954-2008. We found that the combination between a stochastic model for the evolution of the force of mortality and a quantile approach allows to control the deviations of mortality from its expected trend due to the longevity and to limit the probability of underestimation of the survival probabilities within precise limits. The identification of these limits allows the insurance company to adapt its solvency margin to its risk profile. Further research on this subject could be oriented in deepening the topic of the choice of different mortality models in order to quantify the so called model risk.

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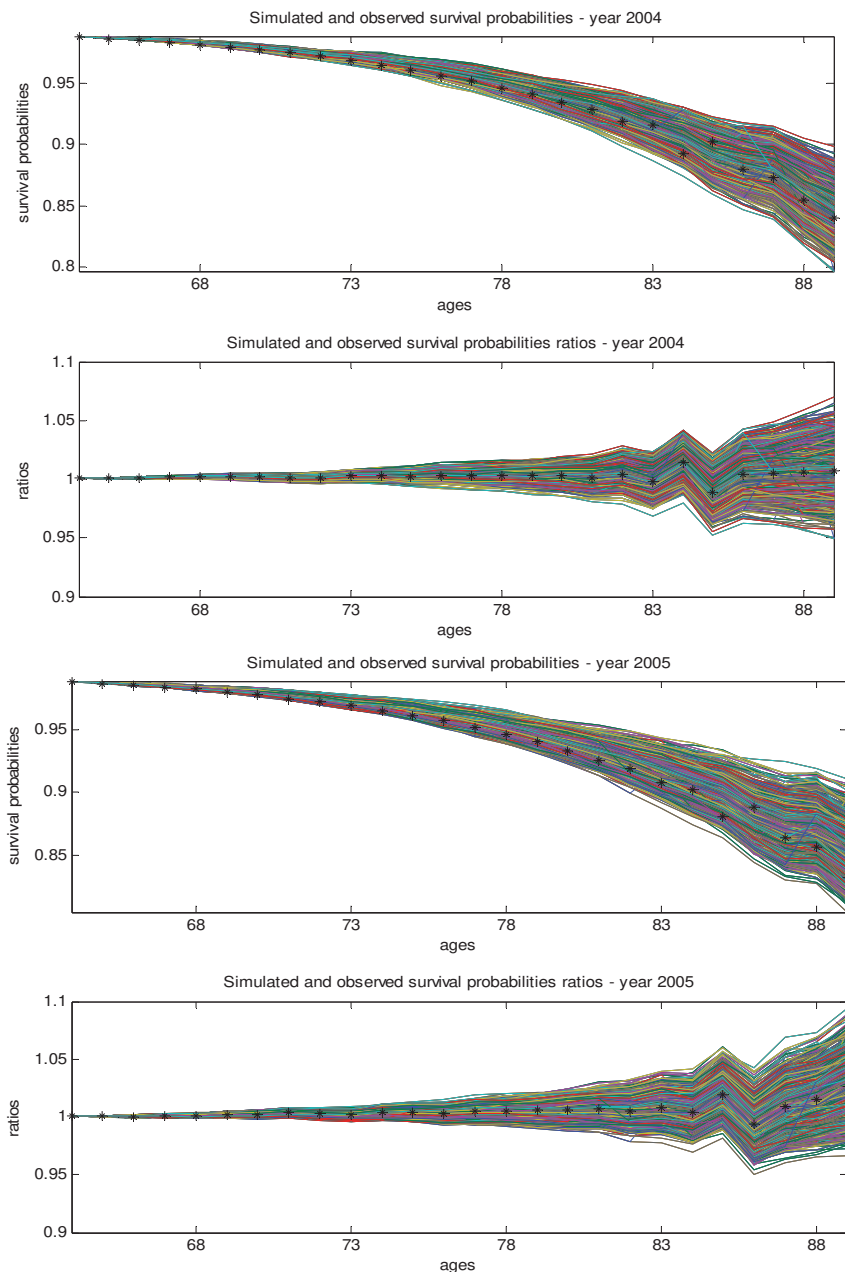
Appendix

Fig. 4a. Annual simulated survival probabilities *Px*(*t*) with *x* ∈ (64.89) for each calendar year *t* ranging from 2004 to 2007. Ratio between annual simulated sruvival probabilities and real data
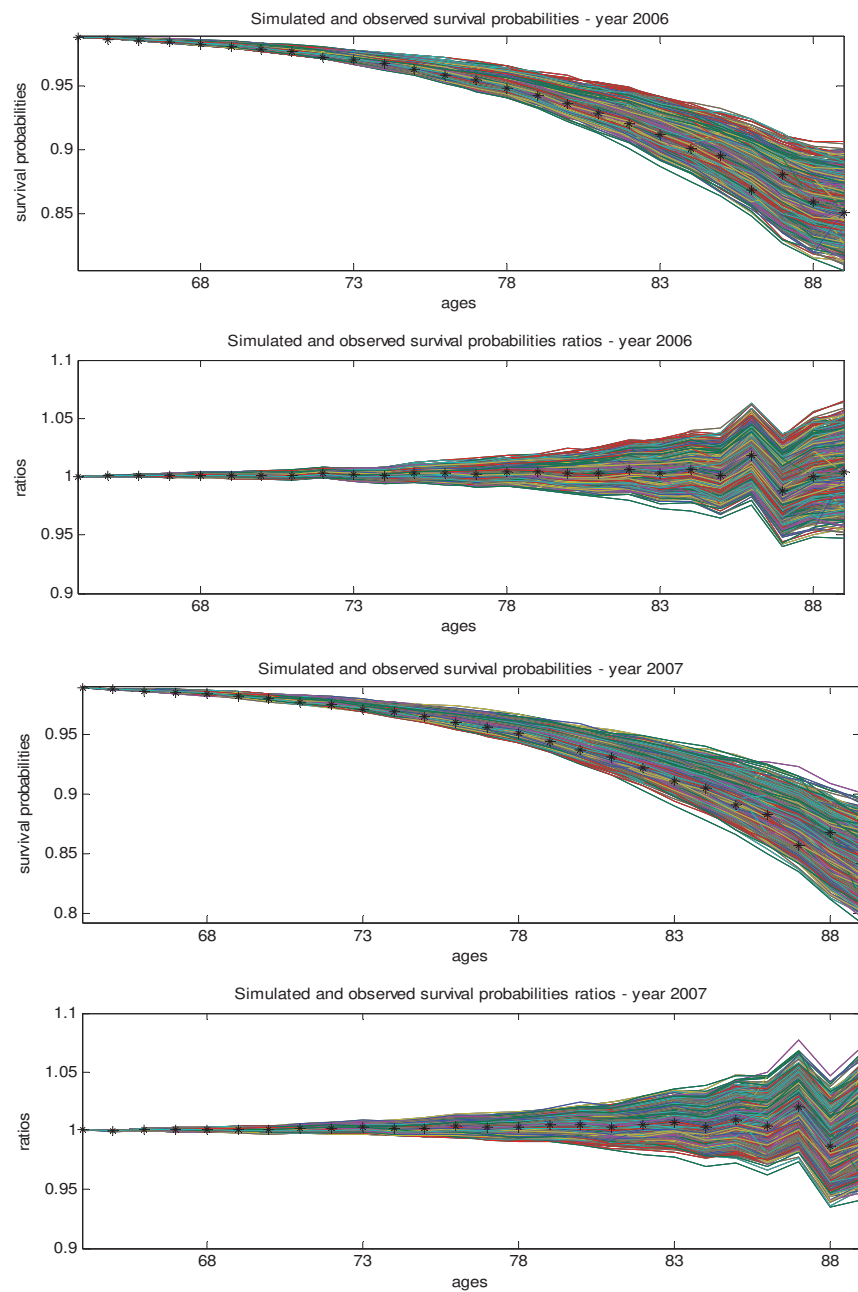
Fig. 4b. Annual simulated survival probabilities $Px(t)$ **with** $x \in (64.89)$ **for each calendar year** t **ranging from 2004 to 2007. Ratio between annual simulated sruvival probabilities and real data**