## "Spectral study of options based on CEV model with multidimensional volatility"

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BUSINESS PERSPECTIVES

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# SPECTRAL STUDY OF OPTIONS BASED ON CEV MODEL WITH MULTIDIMENSIONAL VOLATILITY 


#### Abstract

This article studies the derivatives pricing using a method of spectral analysis, a theory of singular and regular perturbations. Using a risk-neutral assessment, the authors obtain the Cauchy problem, which allows to calculate the approximate price of derivative assets and their volatility based on the diffusion equation with fast and slow variables of nonlocal volatility, and they obtain a model with multidimensional stochastic volatility. Applying a spectral theory of self-adjoint operators in Hilbert space and a theory of singular and regular perturbations, an analytic formula for approximate asset prices is established, which is described by the CEV model with stochastic volatility dependent on $l$-fast variables and $r$-slowly variables, $l \geq 1, r \geq 1, l \in N, r \in N$ and a local variable. Applying the Sturm-Liouville theory, Fredholm's alternatives, as well as the analysis of singular and regular perturbations at different time scales, the authors obtained explicit formulas for derivatives price approximations. To obtain explicit formulae, it is necessary to solve 21 Poisson equations.


## Keywords

CEV model, stochastic multidimensional volatility, spectral theory, singular perturbation theory, regular perturbation theory

## JEL Classification <br> G11, G13, G32

## INTRODUCTION

The constant elasticity of variance (CEV) model is a generalization of geometric Brownian motion models. This model has been introduced by Cox and Ross (1976) for pricing European call options. The CEV model is based on the assumption that the risk-neutral process that describes the stock price $S$ has the form: $d S=(r-q) S d t+\sigma S^{\alpha} d z$, where $r$ risk interest rate, $q$ - dividend yield, $\sigma$ - parameter of volatility, $\alpha$ - positive constant, $d z$ - Wiener process. For $\alpha=1$, the CEV model coincides with the model of the geometric Brownian motion, if $\alpha<1$, then, with the reduction of asset value, its volatility increases, if $\alpha>1$, then, with the increase in the asset value, its volatility also increases. This corresponds to volatility smile and means that volatility is an increasing function depending on the asset price.

Comparing with the models of the geometric Brownian motion, the advantages of a CEV model lie in the fact that the instability ratio correlates with the price of risky assets and may explain empirical bias, such as the volatility smile (Schroder, 1989). The CEV model is usually applied to calculate the theoretical price, sensitivity and expected volatility of options (Emanuel \& MacBeth, 1982; Fouqueet at al., 2000). In recent years, the problem of a pension fund investment is very urgent, it turns out that the CEV model has been successfully applied to study the effective investment strategy (Davydov \& Linetsky, 2001; Davydov \& Linetsky, 2003).

At the end of the last century, the attention of financial scientists was drawn to the problem of relationship between the asset price and its
volatility (Schroder, 1989). It was found that the asset price behaves like volatility. According to a BlackScholes model, volatility is constant (Black \& Scholes, 1973). As a result, this led to a series of works to expand this model. Empirical studies have found that volatility is a random variable dependent on time. Hull and White (1989), Stein and Stein (1991), Heston (1993) introduced analytical models with stochastic volatility. Carr and Linetsky (2006), Aboulaich et al. (2013) investigated the stochastic volatility model with jumps. The CEV model is a generalization of dynamic volatility models. In particular, it provides an opportunity to examine the asset price that changes continuously over time. There is a series of works dedicated to this problem (Andersen \& Piterbarg, 2007; Lindsay \& Brecher, 2012). In this article, we consider the CEV model with multidimensional stochastic volatility. Using a spectral theory of selfadjoint operators in Hilbert space and a theory of singular and regular disturbances, an analytic formula for the approximate asset prices is established, which is described by the CEV model with stochastic volatility dependent on $l$-fast variables and $r$-slowly variables, $l \geq 1, \mathrm{r} \geq 1, l \in N, r \in N$ and a local variable. The theorem of closeness in estimation of financial instruments prices approximation is proved.

## 1. RESEARCH METHODOLOGY

The purpose of the article is to establish the approximate derivative prices, which are defined by CEV model with stochastic multidimensional volatility and depend on many factors of a spectral theory and a theory of perturbations.

Thus, when a CEV model with stochastic multidimensional volatility is adequate for describing the dynamics of an underlying, the spectral method outlined above serves as a powerful tool for analytically pricing derivatives on that underlying. Among the topics that have been addressed by applying spectral methods with multidimensional diffusions are option pricing (both vanilla and exotic), mortgages valuation, interest rate modeling, volatility modeling, and credit risk.

The necessity to have stock price diffusions that don't jump to zero in order to default and still have a non-zero probability of falling to zero leads us to naturally consider CEV processes. Moreover, CEV models have the advantage to provide closed-form formulae for European vanilla options and for the probability of default.

In particular, many problems related to the pricing of derivative assets have been solved analytically by using methods from spectral theory. An overview of the spectral method applied to derivative pricing is as follows.

We consider the probability space $(Q, F, P)$ with systems of Wiener motion
$\left(B^{x}, B^{y_{1}}, \ldots, B^{y_{l}}, B^{z_{1}}, \ldots, B^{z_{r}}\right)$ and an exponential random variable $\eta \sim \operatorname{Exp}(1)$, which is independent of $\left(B^{x}, B^{y_{1}}, \ldots, B^{y_{l}}, B^{z_{1}}, \ldots, B^{z_{r}}\right)$. We consider a homogeneous Markov process, which depends on $l+r+1$ factors $\chi=X, Y_{1}, \ldots, Y_{l}, Z_{1}, \ldots, Z_{r}$, which is defined in some state space $R=H \cdot R^{l} \cdot R^{r}$, where $\left(Y_{1}, \ldots, Y_{l}\right) \in R^{l}, \quad\left(Z_{1}, \ldots, Z_{r}\right) \in R^{r} I \quad$ is the interval in $R$ with points $a_{1}$, and $a_{2}$, such that $-\infty<a_{1}<a_{2}<\infty$. Let that $\chi$ has a beginning at $R$ and instantly disappears as soon as $X$ goes beyond $H$. In particular, the dynamics of $\chi$ with physical measure $P$ is as follows:
$\chi_{t}= \begin{cases}X_{t}, Y_{1 t}, \ldots, Y_{l t}, Z_{1 t}, \ldots, Z_{r t}, & \theta_{H}>t, \\ \Delta, & \theta_{H} \leq t,\end{cases}$ $\theta_{H}=\inf \left(t>0: X_{t} \notin H\right)$,
where $X, Y_{1}, \ldots, Y_{l}, Z_{1}, \ldots, Z_{r}$ are set

$$
\begin{aligned}
& d X_{t}=v\left(X_{t}\right) d t+a\left(X_{t}\right) \times \\
& \times f\left(Y_{1 t}, \ldots, Y_{l t}, Z_{1 t}, \ldots, Z_{r t}\right) d B_{t}^{x}, \\
& d Y_{j t}=\frac{1}{\eta_{j}} \cdot \alpha_{j}\left(Y_{j t}\right) d t+\frac{1}{\sqrt{\eta_{j}}} \cdot \beta_{j}\left(Y_{j t}\right) d B_{t}^{y_{j}}, j=\overline{1, l} . \\
& d Z_{i t}=\gamma_{i} k_{i}\left(Z_{i t}\right) d t+\sqrt{\gamma_{i}} g_{i}\left(Z_{i t}\right) d B_{t}^{z_{i}}, i=\overline{1, r} . \\
& d\left(B^{x}, B^{y_{j}}\right)_{t}=\rho_{x y_{j}} d t, j=\overline{1, l} . \\
& d\left(B^{x}, B^{z_{i}}\right)_{t}=\rho_{x z_{i}} d t, i=\overline{1, r} . \\
& d\left(B^{y_{j}}, B^{z_{i}}\right)_{t}=\rho_{y_{j} z_{i}} d t, j=\overline{1, l}, i=\overline{1, r} .
\end{aligned}
$$

$$
d\left(B^{y_{j}}, B^{y_{r}}\right)_{t}=\rho_{y_{j} y_{s}} d t, \quad j=\overline{1, l}, s=\overline{1, l} .
$$

$$
d\left(B^{z_{i}}, B^{z_{k}}\right)_{t}=\rho_{z_{i} z_{k}} d t, i=\overline{1, n}, \quad k=\overline{1, r}
$$

$\left(X_{0}, Y_{10}, \ldots, Y_{l 0}, Z_{10}, \ldots, Z_{r 0}\right)=$
$=\left(x, y_{10}, \ldots, y_{l 0}, z_{10}, \ldots, z_{r 0}\right) \in R$,
where $\quad \rho_{y_{j} y_{s}}=0, \quad j \neq r, \quad \rho_{z_{i} z_{k}}=0$, $i \neq k \rho_{x y_{j}}, \rho_{x z_{i}}, \rho_{y_{j} z_{i}}$, meet the condition $\left|\rho_{x y_{j}}\right|,\left|\rho_{x z_{i}}\right|,\left|\rho_{y_{j} z_{i}}\right| \leq 1$, and a correlation matrices

$$
\text { of a form: }\left(\begin{array}{ccc}
1 & \rho_{x y_{j}} & \rho_{x z_{i}} \\
\rho_{y_{j} x} & 1 & \rho_{y_{j} z_{i}} \\
\rho_{z_{i} x} & \rho_{z_{i} y_{j}} & 1
\end{array}\right)
$$

semi positively defined, i.e.
$1+2 \cdot \rho_{x y_{j}} \cdot \rho_{x z_{i}} \cdot \rho_{y_{j} z_{i}}-\rho_{x y_{j}}^{2}-\rho_{x z_{i}}^{2}-\rho_{y_{j} z_{i}}^{2} \geq 0$,
$j=\overline{1, l}, i=\overline{1, r}$.
In changing from the physical probability measure to the risk-neutral pricing measure, we consider a class of market prices of risk that is general enough to treat credit, equity, and interest rate derivatives in a single framework.

Process $X$ can represent many economic phenomena and processes, which describe the optimal investment strategies. For example, the stockpiles, the index price, a risk-free shortterm interest rate, etc. Even more broadly, $X$ is an external factor that characterizes the cost of any of the abovementioned processes. We are considering the process $X$ with stochastic volatility $\quad a\left(X_{t}\right) f\left(Y_{1 t}, \ldots, Y_{l t}, Z_{1 t}, \ldots, Z_{n t}\right)>0$, which contains both components: local $a\left(X_{t}\right)$ and nonlocal $f\left(Y_{1 t}, \ldots, Y_{l t}, Z_{1 t}, \ldots, Z_{n t}\right)$. Note that the infinitesimal generators (Infiniti zed) for $Y_{j}$ and $Z_{i}$ have the form $\forall i, j$.
$P_{Y_{j}}^{\eta_{j}}=\frac{1}{\eta_{j}}\left(\frac{1}{2} \cdot \beta_{j}{ }^{2}\left(y_{j}\right) \partial_{y_{j} y_{j}}^{2}+\alpha_{j}\left(y_{j}\right) \partial_{y_{j}}\right)$,
$P_{Z_{i}}^{\delta_{i}}=\gamma_{i}\left(\frac{1}{2} \cdot g_{i}{ }^{2}\left(z_{i}\right) \partial_{z_{i} z_{i}}^{2}+c_{i}\left(z_{i}\right) \partial_{z_{i}}\right)$,
are characterized by the values $1 / \eta_{j}$ and $\gamma_{i}$, respectively. Thus, $Y_{1}, \ldots, Y_{l}$ and $Z_{1}, \ldots, Z_{n}$ have an internal time scale $\eta_{j}>0$ and $1 / \gamma_{i}>0$. We consider $\eta_{j}<1$ and $\gamma_{i}<1$, so that the internal time scale $Y_{j}$ is small, and the internal time scale $Z_{i}$ is large. Consequently, $Y_{j}, j=\overline{1, l}$ are fast variables, and $Z_{i}, i=\overline{1, n}$ are slowly variables. Note that $P_{Y_{j}}^{\eta_{j}}$ and $P_{Z_{i}}^{\delta_{i}}$ have the form

$$
\begin{aligned}
& L=\frac{1}{2} \cdot a^{2}(x) \partial_{x x}^{2}+b(x) \partial_{x}-k(x) \\
& x \in\left(a_{1}, a_{2}\right), k(x)=0
\end{aligned}
$$

for all $x \in I$, are always self-conjugated in a Hilbert space $H=L^{2}(h, p)$, where $H \in R$ is the interval terminating at $a_{1}$ and $a_{2}$ and $p$ is a diffusion density rate.

The boundary conditions for $a_{1}$ and $a_{2}$ are implemented on the output, input and regular bounds.

We evaluate the derivative security with payoff at time $t>0$, which may depend on trajectory $X$. In particular, we will consider the forms of payoff:

Payoff $=H\left(X_{t}\right) \cdot I_{\theta>t}$,
where $\theta$ is a random moment of time during which there is a failure to make a payment of premium. Since we are interested in derivatives estimates, we must determine the dynamics $\left(X, Y_{1}, \ldots, Y_{l}, Z_{1}, \ldots, Z_{r}\right)$ with risk-neutral measure estimate, which we denote as $\widetilde{P}$. We have the following dynamics:

$$
\begin{aligned}
& \left(d X_{t}=\binom{b\left(X_{t}\right)-a\left(X_{t}\right) \cdot f\left(Y_{1 t}, \ldots, Y_{t t}, Z_{1 t}, \ldots, Z_{r t}\right) \times}{\times Q\left(Y_{1 t}, \ldots, Y_{l t}, Z_{1 t}, \ldots, Z_{r t}\right)} d t+\right. \\
& +a\left(X_{t}\right) \cdot f\left(Y_{1 t}, \ldots, Y_{l t}, Z_{1 t}, \ldots, Z_{r t}\right) \tilde{d}_{t}^{x}, \\
& d Y_{j t}=\binom{\frac{1}{\eta_{j}} \cdot \alpha_{j}\left(Y_{j t}\right)-\frac{1}{\sqrt{\eta_{j}}} \cdot \beta_{j}\left(Y_{j t}\right) \times}{\times \Phi_{j}\left(Y_{1 t}, \ldots, Y_{l t}, Z_{1 t}, \ldots, Z_{r t}\right)} d t+\frac{1}{\sqrt{\eta_{j}}} \cdot \beta_{j}\left(Y_{j t}\right) \widetilde{d B_{t}^{y_{j}}}, \\
& d Z_{i t}=\binom{\gamma_{i} k_{i}\left(Z_{i t}\right)-\sqrt{\gamma_{i}} g_{i}\left(Z_{i t}\right) \times}{\times \Psi_{i}\left(Y_{1 t}, \ldots, Y_{l t}, Z_{1 t}, \ldots, Z_{r t}\right)} d t+\sqrt{\gamma_{i}} g_{i}\left(Z_{i t}\right) \widetilde{d B_{t}}, \\
& \left\{d\left\langle\widetilde{B}^{x}, \widetilde{B}^{y_{j}}\right\rangle_{t}=\rho_{x y_{j}} d t, j=\overline{1, l}\right. \text {. } \\
& d\left\langle\widetilde{B}^{x}, \widetilde{B}^{z_{i}}\right\rangle_{t}=\rho_{x z_{i}} d t, i=\overline{1, r} \text {. } \\
& d\left\langle\widetilde{B}^{y_{j}}, \widetilde{B}^{z_{i}}\right\rangle_{t}=\rho_{y_{j} z_{i}} d t, j=\overline{1, l}, i=\overline{1, r} . \\
& d\left\langle\widetilde{B}^{y_{j}}, \widetilde{B}^{v_{s}}\right\rangle_{t}=\rho_{y_{j} y_{s}} d t, j=\overline{1, l}, s=\overline{1, l} . \\
& d\left\langle\widetilde{B}^{z_{i}}, \widetilde{B}^{z_{k}}\right\rangle_{t}=\rho_{z_{i z_{k}}} d t, i=\overline{1, n}, k=\overline{1, n} \text {. } \\
& \left(X_{0}, Y_{10}, \ldots, Y_{10}, Z_{10}, \ldots, Z_{r 0}\right)= \\
& =\left(x, y_{10}, \ldots, y_{10}, z_{10}, \ldots, z_{r 0}\right) \in E \text {, }
\end{aligned}
$$

where
$\tilde{d B}_{t}^{x}:=d B_{t}^{x}+\binom{\frac{v\left(X_{t}\right)-b\left(X_{t}\right)}{a\left(X_{t}\right) \cdot f\left(Y_{1 t}, \ldots, Y_{t t}, Z_{1 t}, \ldots, Z_{n t}\right)}}{+Q\left(Y_{1 t}, \ldots, Y_{t}, Z_{1 t}, \ldots, Z_{r t}\right)} d t$,
$\widetilde{d B_{t}^{y_{j}}}:=d B_{t}^{v_{j}}+\Phi_{j}\left(Y_{1 t}, \ldots, Y_{l t}, Z_{1 t}, \ldots, Z_{r t}\right) d t$,
$\widetilde{d B_{t}}:=d B_{t}^{z_{i}}+\Psi_{i}\left(Y_{1 t}, \ldots, Y_{l t}, Z_{1 t}, \ldots, Z_{r t}\right) d t$,
where $\rho_{y, y_{s}}=0, j \neq s, \rho_{z i z_{k}}=0, i \neq k$.
We implement such conditions so that the system (1) has the only strong solution.

Random time $\tau$ is the time of a derivative asset. In our case, a default can occur in neither of two ways:
when $X$ is beyond the interval $H$;
at random time $\theta_{h}$, which is managed by risk level $h\left(X_{t}\right) \geq 0$.

This can be expressed as follows:
$\left\{\begin{array}{l}\theta=\tau_{H} \cup \tau_{h}, \\ \theta_{H}=\inf \left\{t \geq 0: X_{t} \notin H\right\}, \\ \theta_{H}=\inf \left\{\begin{array}{l}t \geq 0: \int_{0}^{t} h\left(X_{s}\right) d s \geq \\ \geq \varepsilon\left(X, Y_{1}, \ldots, Y_{l}, Z_{1}, \ldots, Z_{n}\right)\end{array}\right\}\end{array}\right.$
Note that the random variable $\eta$ is independent of $X, Y_{1}, \ldots, Y_{l}, Z_{1}, \ldots, Z_{n}$.

We will calculate the derivative asset of some payoff using risk-free pricing and Markovian chain $X$, the price $w^{\bar{\eta},}\left(t, x, y_{1}, \ldots, y_{l}, z_{1}, \ldots, z_{r}\right)$ of some derivative assets at the initial moment of time has the form:
$w^{\bar{\eta}, \bar{\gamma}^{\prime}}\left(t, x, y_{1}, \ldots, y_{l}, z_{1}, \ldots, z_{r}\right)=$
$=\widetilde{E}_{x, y_{1}, \ldots, v_{l}, z_{1}, \ldots z_{r}}\left[\exp \left(-\int_{0}^{t} r\left(X_{m}\right) d m\right) H\left(X_{t} I_{\{>\theta \theta\}}\right)\right]$,
where $\bar{\eta}=\left(\eta_{1}, \ldots, \eta_{l}\right), \quad \overline{\gamma^{\prime}}=\left(\gamma_{1}, \ldots, \gamma_{r}\right)$, and $\left(x, y_{1}, \ldots, y_{l}, z_{1}, \ldots, z_{r}\right) \in E$ is a starting point of the process $\left(X, Y_{1}, \ldots, Y_{l}, Z_{1}, \ldots, Z_{r}\right)$ applying the Feynman-Kats formula, we can show that $w^{\overline{\eta^{,}} \bar{\gamma}^{\prime}}\left(t, x, y_{1}, \ldots, y_{l}, z_{1}, \ldots, z_{r}\right) \quad$ satisfies the following Cauchy problem (Linetsky, 2007):
$\left(-\partial_{t}+P^{\bar{\eta}, \overline{\gamma^{\prime}}}\right) w^{\bar{\eta}, \overline{\gamma^{\prime}}}=0$,
$\left(y_{1}, \ldots, y_{l}, z_{1}, \ldots, z_{r}\right) \in R, \quad t \in R^{+}$,
$w^{\bar{\eta}, \bar{r}^{\prime}}\left(0, x, y_{1}, \ldots, y_{l}, z_{1}, \ldots, z_{r}\right)=H(x)$,
where the operator $P^{\bar{\eta}, \overline{\gamma^{\prime}}}$ has the form:
$P^{\bar{\eta}, \overline{\gamma^{\prime}}}=\sum_{j=1}^{l} \frac{1}{\eta_{j}} P_{0 j}+\sum_{j=1}^{l} \frac{1}{\sqrt{\gamma_{j}}} P_{1 j}+$
$+P_{2 j}+\sum_{i, j} \sqrt{\frac{\gamma_{i}}{\eta_{j}}} U_{3 i j}+\sum_{i} \sqrt{\gamma_{i}} U_{1 i}+\sum_{i} \gamma_{i} U_{2 i}$,
$P_{0 j}=\frac{1}{2} \cdot \beta_{j}^{2}\left(y_{j}\right) \partial_{y_{j} y_{j}}^{2}+\alpha_{j}\left(y_{j}\right) \partial_{y_{j}}, \quad j=\overline{1, l}$,
$P_{1 j}=\beta_{j}\left(y_{j}\right)\binom{\rho_{x y_{j}} a(x) f\left(y_{1}, \ldots, y_{l}, z_{1}, \ldots, z_{r}\right) \partial_{x}-}{-\Phi_{j}\left(y_{1}, \ldots, y_{l}, z_{1}, \ldots, z_{r}\right)} \partial_{y_{j}}$,
$P_{2 j}=\frac{1}{2} \cdot a^{2}(x) f^{2}\left(y_{1}, \ldots, y_{l}, z_{1}, \ldots, z_{r}\right) \partial_{x x}^{2}+$
$+\binom{b(x)-a(x) Q\left(y_{1}, \ldots, y_{l}, z_{1}, \ldots, z_{r}\right) \times}{\times f\left(y_{1}, \ldots, y_{l}, z_{1}, \ldots, z_{r}\right)} \partial_{x}-k(x)$,
$U_{3 i j}=\rho_{x z_{i}} \beta_{j}\left(y_{j}\right) g_{i}\left(z_{i}\right) \partial_{y_{j} z_{i}}^{2}$,
$U_{1 i}=g_{i}\left(z_{i}\right)\binom{\rho_{x z_{i}} a(x) f\left(y_{1}, \ldots, y_{l}, z_{1}, \ldots, z_{r}\right) \partial_{x}-}{-\Psi_{i}\left(y_{1}, \ldots, y_{l}, z_{1}, \ldots, z_{r}\right)} \partial_{z_{i}}$,
$U_{2 i}=\frac{1}{2} \cdot g_{i}^{2}\left(z_{i}\right) \partial_{z_{i} z_{i}}^{2}+c_{i}\left(z_{i}\right) \partial_{z_{i}}$,
$k(x)=r(x)+h(x), \quad P_{0 j}=P_{Y_{j}}^{1}$.
We assume that the diffusion with the infinitesimal generator $P_{Y_{j}}^{1}$ has an invariant distribution $H$ with density $\pi_{j}\left(y_{j}\right)$.
$\pi_{j}\left(y_{j}\right)=\frac{2}{\beta_{j}^{2}\left(y_{j}\right)} \exp \left\{\int_{y_{j 0}}^{y_{j}} \frac{2 \alpha_{j}(\theta)}{\beta_{j}^{2}(\theta)} d \theta\right\}, \quad \forall j=\overline{1, l}$.
Besides the initial condition (3), function $w^{\bar{\eta}, \bar{\gamma}^{\prime}}\left(t, x, y_{1}, \ldots, y_{l}, z_{1}, \ldots, z_{r}\right)$ must meet boundary conditions at the points $a_{1}$ and $a_{2}$ of interval $H$. The boundary conditions at points $a_{1}$ and $a_{2}$ belong to domain $P^{\bar{\eta}, \gamma^{\prime}}$ and will depend on the nature of process $X$ on the points of $H$ and are classified as natural, output, input or regular (Borodin \& Salminen, 2002). The Cauchy problem (1)-(2) for

$$
\binom{f, \alpha_{1}, \ldots, \alpha_{l}, \beta_{1}, \ldots, \beta_{r}, \Phi_{1}, \ldots, \Phi_{l}, c_{1}, \ldots, c_{r}}{g_{1}, \ldots, g_{r}, \Psi_{1}, \ldots, \Psi_{r}}
$$

has no analytical solution. However, for fixed $\overline{\gamma^{\prime}}$, conditions containing $\bar{\eta}$ and deviate arbitrarily small in $\bar{\eta}$ axis, which leads to singular perturbations.

For fixed $\eta_{j}$ conditions containing $\gamma_{i}$ are small for some small $\overline{\gamma^{\prime}}$-axis, which causes regular perturbations. Thus, the $\bar{\eta}$-axis and $\overline{\gamma^{\prime}}$-axis initiate a combined singular-regular perturbation of the operator $P_{2}$. In order to find the asymptotic solution of the Cauchy problem (2)-(3), we develop $w^{\bar{\eta}, \gamma^{\prime}}$ in orders $\sqrt{\eta_{j}}$ and $\sqrt{\gamma_{i}}$ :

$$
\begin{aligned}
& w^{\bar{\eta}, \bar{\gamma}^{\prime}}=\sum_{i_{1} \geq 0} \ldots \sum_{i_{i} \geq 0} \sum_{j_{i} \geq 0} \ldots \sum_{j_{r} \geq 0}{\sqrt{\eta_{1}}}^{j_{1}} \ldots \\
& \ldots{\sqrt{\eta_{l}}}_{j_{l}}^{\gamma_{1}}{\sqrt{i_{1}}}^{i_{r}} \\
& \boldsymbol{\gamma}_{r} \\
& w_{j_{1}, \ldots, j_{n}, i_{1}, \ldots i_{l}}
\end{aligned},
$$

where

$$
\begin{aligned}
& \sum_{i_{1} \geq 0} \ldots \sum_{i_{i} \geq 0} \sum_{j_{1} \geq 0} \ldots \sum_{j_{r} \geq 0}{\sqrt{\eta_{1}}}^{j_{1}} \cdots{\sqrt{\eta_{l}}}^{j_{l}}{\sqrt{\gamma_{1}}}^{i_{1}} \cdots \\
& \ldots{\sqrt{\gamma_{r}}}^{i_{r}} w_{j_{1}, \ldots, j_{n}, i_{1}, \ldots i_{l}}=\lim _{m_{1} \rightarrow \infty, \ldots, m_{l+r} \rightarrow \infty} \sum_{i_{1} \geq 0}^{m_{1}} \ldots \sum_{i_{1} \geq 0}^{m_{l} \geq 0} \sum_{j_{1} \geq 0}^{m_{l+1} \geq 0} \ldots \\
& \ldots \sum_{j_{n} \geq 0}^{m_{l+r}} \ldots{\sqrt{\eta_{l}}}^{j l}{\sqrt{\gamma_{1}}}^{i_{1}} \ldots{\sqrt{\gamma_{r}}}^{i_{r}} w_{j_{1}, \ldots, j_{r} i_{1}, \ldots i_{l}},
\end{aligned}
$$

Approximate price is calculated as follows:

$$
w^{\bar{\eta}, \bar{\gamma}^{\prime}} \approx w_{\overline{0}, \overline{o^{\prime}}}+\sum_{j=1}^{l} \sqrt{\eta_{j}} w_{\overline{1}_{j}, \overline{0}^{\prime}}+\sum_{i=1}^{r} \sqrt{\gamma_{i}} w_{\overline{0}, \overline{1_{i}}} .
$$

The choice of development in half-integer orders $\eta_{j}$ and $\gamma_{i}$ is natural for $P^{\bar{\eta}, \bar{\gamma}^{\prime}}$.

By carrying out an analysis of singular perturbations at corresponding levels, we obtain that $w_{\overline{0}, \overline{\sigma^{\prime}}}, \quad w_{\overline{1_{j}}, \overline{0^{\prime}}}, \quad w_{\overline{0}, \overline{1}, \bar{i}}$ does not depend on $y_{1}, \ldots, y_{l}$. The main results of asymptotic analysis are given using the following formulae:

$$
\begin{align*}
& \sum_{j=1} P_{0 j} w_{\overline{\bar{z}_{j}, \overline{0}^{\prime}}}+\left(-\partial_{t}+\left\langle P_{2}\right\rangle\right) w_{\overline{0}, \overline{o^{\prime}}}=0,  \tag{3}\\
& w_{\overline{0} \overline{0^{\prime}}}\left(0, x, z_{1}, \ldots, z_{r}\right)=H(x)
\end{align*}
$$

$P_{0 j} w_{\overline{3_{j}}, \overline{0}^{\prime}}+P_{1 j} w_{\overline{\overline{2}_{j}}, \overline{0}^{\prime}}+\left(-\partial_{t}+\left\langle P_{2}\right\rangle\right) w_{\overline{1}_{j}, \overline{0^{\prime}}}+$
$+\sum_{k \neq j} P_{1 k} w_{\overline{\mathrm{T}}_{k j}, \overline{0^{\prime}}}+\sum_{i \neq j} P_{1 i}=$
$=\mathcal{W}_{j} w_{\overline{0}, \overline{0^{\prime}}}, w_{\overline{1}_{j}, \overline{0}^{\prime}}\left(0, x, z_{1}, \ldots, z_{n}\right)=0$,
$\overline{1}_{k j}=(\underbrace{0, \ldots}_{k} \underbrace{10,1,0, \ldots .0}_{j})$.

From the analysis of regular perturbations, we obtain
$\left(-\partial_{t}+\langle P\rangle_{2}\right) w_{\overline{0}, \overline{1_{i}^{\prime}}}=v_{i} \partial_{z_{i}} w_{\overline{0}, \overline{0_{0}}}$,
$w_{\overline{0}, \overline{1_{i}^{\prime}}}\left(0, x, z_{1}, \ldots, \mathrm{z}_{r}\right)=0, i=\overline{1, r}$.
Operators $\left\langle P_{2}\right\rangle, \quad w_{j}, \quad v_{i}$ та $\partial_{z_{i}}$ are set by the formulae:
$\left\langle P_{2}\right\rangle=\frac{1}{2} \cdot \bar{\sigma}^{2} a^{2}(x) \partial_{x x}^{2}+(b(x)-\overline{f Q} a(x)) \partial_{x}-k(x)$, $x \in\left(a_{1}, a_{2}\right)$,
$\mathcal{W}_{j}=-v_{3 j} a(x) \partial_{x} a^{2}(x) \partial_{x x}^{2}-v_{2 j} a^{2}(x) \partial_{x x}^{2}-$
$-\mathcal{U}_{2 j} a(x) \partial_{x} a(x) \partial_{x}-\mathcal{U}_{1 j} a(x) \partial_{x}$,
$V_{i}=-i_{1 i} a(x) \partial_{x}-v_{0 i}$ and $\partial_{z_{i}}=\partial_{z_{i}} \bar{\sigma}_{\bar{\sigma}}+\overline{f Q} \partial^{\prime} \partial_{\overline{f Q}}$,
$v_{1 i}:=g_{i} \rho_{x z_{i}}\langle f\rangle, v_{0}=g_{i}\left\langle\Psi_{i}\right\rangle$,
$\forall i=\overline{1, n}$ and the norm function
$\langle\mathcal{X}\rangle_{j}:=\int \mathcal{X}\left(y_{1}, \ldots y_{l}\right) \pi_{j}\left(y_{j}\right) d y_{j}, \quad \forall j=\overline{1, l}$,
$\langle\mathcal{X}\rangle_{1,2}=\int_{R^{2}} \mathcal{X}\left(y_{1}, \ldots y_{l}\right) \pi_{1}\left(y_{1}\right) \pi_{2}\left(y_{2}\right) d y_{1} d y_{2}, \ldots\langle\mathcal{X}\rangle_{l-1, l}=$
$=\int_{R^{I}} \mathcal{X}\left(y_{1}, \ldots y_{l}\right) \pi_{1}\left(y_{1}\right) \ldots \pi_{l}\left(y_{l}\right) d y_{1} \ldots d y_{l}$,
$\langle\mathcal{X}\rangle_{l-1, l}=\mathcal{X}, \quad\langle f Q\rangle:=\overline{f Q}, \quad\left\langle f^{2}\right\rangle=\bar{\sigma}^{2}$.
We find solutions to the equations (3)-(5) on the basis of Eigen functions, eigenvalues of the operator $\left\langle P_{2}\right\rangle$, each of which satisfies a corresponding Poisson equation:

$$
\begin{aligned}
& P_{01} v_{1}=f^{2}-\left\langle f^{2}\right\rangle_{1} \\
& P_{02} v_{2}=\left\langle f^{2}\right\rangle_{1}-\left\langle f^{2}\right\rangle_{1,2}, \ldots \\
& \ldots P_{0 l} v_{l}=\left\langle f^{2}\right\rangle_{l-2, l-1}-\left\langle f^{2}\right\rangle_{l-1, l}
\end{aligned}
$$

Theorem 1: Let the equation be:
$-\left\langle P_{2}\right\rangle v_{n}=\mu_{n} v_{n}, v_{n} \in \operatorname{dom}\left(\left\langle P_{2}\right\rangle\right)$,
and $h \in H$. Then, the solution $w_{\overline{0}, \overline{\bar{D}^{\prime}}}$ has the form:
$w_{\overline{0}, \overline{T_{0}}}=\sum_{n=1}^{\infty} k_{n} v_{n} T_{n}, k_{n}=\left(v_{n}, h\right), T_{n}=e^{-t \mu_{n}}$.
Theorem 2: Let $k_{n}, v_{n}, T_{n}$ be equations (6)-(7):
$\mathcal{W}_{j k, n}:=\left(v_{k}, \mathcal{W}_{j} v_{n}\right), M_{k, n}:=\frac{T_{k}-T_{n}}{\mu_{k}-\mu_{n}}$.
Then, the solution $w_{\overline{\mathrm{T}}_{1}, \overline{0}^{\text {o }}}$ of equation (4) has the form:
$w_{\bar{i}, \overline{V^{\prime}}}=\sum_{n} \sum_{k \neq n} k_{n} \mathcal{W}_{j k, n} \nu_{k} M_{k, n}-\sum_{n} k_{n} \mathcal{W}_{j n, n} \nu_{n} t T_{n}$.
Note that $w_{\mathrm{i}_{1}, \overline{0}^{\prime}}$ is linear in the parameter group $\left(\vartheta_{3 j}, \vartheta_{2 j}, u_{2 j}, u_{1 j}\right)$.

Theorem 3: Let $k_{n}, v_{n}$ and $T_{n}$ are set by (6)-(7) and $M_{k, n}$ from (8)-(9), then:
$\tilde{\mathcal{V}}_{i k, n}:=\left(v_{k}, \mathcal{V}_{i} \partial_{z_{i}} v_{n}\right), V_{i k, n}:=\left(v_{k}, \nu_{i} v_{n}\right)$,
$M_{i k, n}:=\frac{T_{k}-T_{n}}{\left(\mu_{k}-\mu_{n}\right)^{2}}+\frac{t T_{n}}{\mu_{k}-\mu_{n}}$.
Then, the solution $w_{\overline{0}, \bar{T}_{i}}$ has the form:
$w_{\overline{0}, \bar{T}_{i}}=\sum_{n} \sum_{k \neq n} k_{n} \tilde{\nu}_{i k, n} v_{k} M_{i k, n}-\sum_{n} k_{n} \tilde{v}_{i n, n} v_{n} t T_{n}+$
$\left.+\sum_{n} \sum_{k \neq n}\left(\partial_{z_{i}} k_{n}\right) \nu_{i k, n} v_{k} M_{i k, n}-\sum_{n}\left(\partial_{z_{i}} k_{n}\right)\right)_{i n, n} v_{n} t T_{n}+$
$+\sum_{n} \sum_{k \neq n} k_{n} v_{i k, n} v_{k}\left(\partial_{z_{i}} \mu_{n}\right) N_{i k, n}-$
$-\sum_{n} k_{n} v_{i n, n} v_{n}\left(\partial_{Z_{i}} \mu_{n}\right) \frac{1}{2} t^{2} T_{n}$.
We proved that $u_{\overline{0}, \overline{l_{i}^{\prime}}}$ are linear in
$\left(v_{1 i} \bar{\sigma}^{\prime}, b_{1 i} \overline{f Q^{\prime}}, b_{0 i} \bar{\sigma}^{\prime}, b_{0 i} \overline{f Q^{\prime}}\right)$.
We have obtained the approximate solution:
for the valuation of derivative assets. For a more accurate result, we assume that the payoff function $H(x)$ and its derivative are smooth and limited functions. Thus, we restrict our analysis of derivatives to a smooth and limited payoff; in
this case, the closeness estimates is based on the following theorem.

Theorem 4: If there is fixed ( $t, x, y_{1}, \ldots, y_{l}, z_{1}, \ldots, z_{r}$ ), then there exists a $C$ constant such that for all $\eta_{j} \leq 1, \gamma_{i} \leq 1$, the following inequality takes place:
$\left|w^{\bar{\eta}, \overline{\gamma^{\prime}}}-\left(w_{\overline{0}, \overline{o^{+}}}+\sum_{j=1}^{l} \sqrt{\eta_{j}} w_{\overline{1}_{j}, \overline{0^{\prime}}}+\sum_{i=1}^{n} \sqrt{\gamma_{i}} w_{\overline{0}, \overline{i_{i}}}\right)\right| \leq$
$\leq C\left(\sum_{j=1}^{l} \varepsilon_{j}+\sum_{i=1}^{r} \delta_{i}\right)$.
Theorem 4 provides us information on how the approximate price behaves when $\eta_{j} \rightarrow 0$ and $\gamma_{i} \rightarrow 0$.

## 2. APPLICATION OF THE DESCRIBED METHODOLOGY

The models developed by these scholars have their advantages and disadvantages, but each of them is used to increase the liquidity of financial markets. The findings are credit spread of credit market instruments, calculating option prices for interest rates, determining the risk and derivatives' rate of return of the stock market financial instruments.

Using the method of Eigen function expansions, we derive analytical solutions for zero-coupon bonds and bond options under CEV processes for the shadow rate. This class of models can be used to model low interest rate regimes.

Let's assume that the asset is defined by $S_{t}=I_{\{\theta>t\}} X_{t}$. Since $S$ must be positive, the space of states $X$ will be $\left(a_{1}, a_{2}\right)=(0, \infty)$. A multiscale diffusion is formed on the default leap using a method of continuous variations (Carr \& Linetsky, 2006). In particular, $\widetilde{P}$ is the dynamics of $X$ before default is set:
$d X_{t}=\left(v+b X_{t}^{2 \kappa}\right) X_{t} d t+$
$+\left(f\left(Y_{1}, \ldots, Y_{l}, Z_{1}, \ldots, Z_{n}\right) X_{t}^{\kappa}\right) X_{t} \tilde{d B}_{t}^{x}$,
$h\left(X_{t}\right)=v+b X_{t}^{2 k}$.
For ease of calculation, the risk-free interest rate is zero: $r=0$. Let's calculate the approximate price of a European option, which is described by $S$.

We write down the operator $\left\langle P_{2}\right\rangle$ and associated with it densities at a rate $p(x)$ :
$\left\langle P_{2}\right\rangle=\frac{1}{2} \cdot \bar{\sigma}^{2} x^{2 \kappa+2} \partial_{x x}^{2}+\left(v+b x^{2 \kappa}\right) x \partial_{x}-\left(v+b x^{2 \kappa}\right)$,
$p(x)=\frac{2}{\bar{\sigma}^{2}} \cdot x^{\frac{2 b}{\overline{\sigma^{2}}-2-2 \kappa}} \exp \left(W x^{-2 \kappa}\right), W=\frac{v}{\bar{\sigma}^{2}|\kappa|}$.
We consider the diffusion operator (10), the end of the interval $a_{2}=\infty$ is a natural boundary. However, the classification of end point $a_{1}=0$ depends on the value of $k$ and $\frac{b}{\bar{\sigma}^{2}}$, i.e.:

1) $\frac{b}{\bar{\sigma}^{2}} \geq \frac{1}{2}, k<0, a_{1}=0$ is a trivial case;
2) $\frac{b}{\bar{\sigma}^{2}} \in\left(0, \frac{1}{2}\right), \quad k \in\left[\frac{b}{\bar{\sigma}^{2}}-\frac{1}{2}, 0\right), \quad a_{1}=0 \quad$ this
number plays a role of a starting point;
3) $\frac{b}{\bar{\sigma}^{2}} \in\left(0, \frac{1}{2}\right), \quad k<\frac{b}{\bar{\sigma}^{2}}-\frac{1}{2}, \quad a_{1}=0$ at such a condition, the start of the interval is constant.

If $\quad(b, \bar{\sigma}, k) \quad$ satisfy $\quad \frac{b}{\bar{\sigma}^{2}} \in\left(0, \frac{1}{2}\right), \quad$ and $k \in\left[\frac{b}{\bar{\sigma}^{2}}-\frac{1}{2}, 0\right), a_{1}=0$ then $a_{1}$ is considered as a keeling boundary. In this case, we find Eigen functions and eigenvalues (7) boundary conditions:
$\lim _{x \rightarrow 0} v_{n}=0$, if $\frac{b}{\bar{\sigma}^{2}} \in\left(0, \frac{1}{2}\right)$.
The solution has the following form (MendozaArriaga et al., 2010):
$v_{n}=W^{\frac{\nu}{2}} \sqrt{\frac{(n-1)!v}{\Gamma(v+n)}} x \exp \left(-W x^{-2 \kappa}\right) L_{n-1}^{(\nu)}\left(W x^{-2 \kappa}\right)$,
$n=1,2,3, \ldots$,
$\mu_{n}=2 v|\kappa|(n+v), v=\frac{1+2\left(\frac{2}{\bar{\sigma}^{2}}\right)}{2|\kappa|}$,
where $L_{n}^{(\nu)}$ are generalized Laguerre polynomials. For the system (10), operators $\mathcal{W}_{j}$ and $\mathcal{V}_{j}$ have the form:
$\mathcal{W}_{j}=-v_{j 3} \kappa^{\kappa+1} \partial_{x} x^{2 \kappa+2} \partial_{x x}^{2}-v_{j 2} x^{2 \kappa+2} \partial_{x x}^{2}$,
$\mathcal{V}_{j}=-V_{j 1} x^{\kappa+1} \partial_{x}-v_{j 0}$.
Payoff for a European call option with the execution price $K>0$ can be expanded as follows:

$$
\begin{equation*}
\left(K-S_{t}\right)^{+}=\left(K-X_{t}\right)^{+} I_{\{\theta>t\}}+K\left(1-I_{\{\theta>t\}}\right) \tag{11}
\end{equation*}
$$

The first item on the right-hand side (11) is the option payoff before the default at time $t$. The second item represents option pay off after the default, which may occur at time $t$. Thus, the option value with execution price $K$ is designated by $w^{\bar{\eta}, \overline{\gamma^{\prime}}}(t, x ; K)$ can be expressed as a sum of two parts:
$w^{\bar{\eta}, \overline{\gamma^{\prime}}}(t, x ; K)=w_{0}^{\bar{\eta}, \overline{\gamma^{\prime}}}(t, x ; K)+w_{D}^{\bar{\eta}, \overline{\gamma^{\prime}}}(t, x ; K)$,
where

$$
\begin{aligned}
& w_{0}^{\bar{\eta}, \bar{\gamma}^{\prime}}(t, x ; K)=\widetilde{E}_{x, \mathrm{y}_{1}, \ldots, \mathrm{y}_{1}, \mathrm{z}_{1}, \ldots, \mathrm{z}_{\mathrm{r}}}\left[\left(K-X_{t}\right)^{+} I_{\{\theta>t\}}\right], \\
& w_{D}^{\bar{\eta}, \bar{\gamma}^{\prime}}(t, x ; K)=K-K \widetilde{K E}_{x, \mathrm{y}_{1}, \ldots, \mathrm{y}_{1}, z_{1}, \ldots, \mathrm{z}_{\mathrm{r}}}\left[I_{\{\theta>t\}}\right]= \\
& K-K \int_{0}^{\infty} E_{x, \mathrm{y}_{1}, \ldots, \mathrm{y}_{1}, \mathrm{z}_{1}, \ldots, \mathrm{z}_{\mathrm{r}}}\left[\delta_{x^{\prime}}\left(X_{t}\right) I_{\{\theta>t\}}\right] d x^{\prime}= \\
& =K-K \int_{0}^{\infty} w_{1}^{\bar{\eta}, \overline{\gamma^{\prime}}}\left(t, x ; x^{\prime}\right) d x^{\prime}, \\
& w_{1}^{\bar{\eta}, \overline{\gamma^{\prime}}}\left(t, x ; x^{\prime}\right)=\widetilde{E}_{x, \mathrm{y}_{1}, \ldots, \mathrm{y}_{1}, \mathrm{z}_{1}, \ldots, \mathrm{z}_{\mathrm{r}}}\left[\delta_{x^{\prime}}\left(X_{t}\right) I_{\{\theta>t\}}\right] .
\end{aligned}
$$

Note that $1 \notin L^{2}\left(R^{+}, p\right)$, we used the fact that the integral of Dirac's function is equation. As payoff functions:
$H_{0}(x)=(K-x)^{+}$and
$H_{1}(x)=\delta_{x^{\prime}}(x) 4_{x^{\prime}} \in L^{2}\left(R^{+}, p\right)$, we can calculate:
$k_{0, n}=\left(v_{n}(\cdot),(K-\cdot)^{+}\right), k_{1, n}=\left(v_{n}, \delta_{x^{\prime}}\right)$.
The coefficients can be found in Mendoza-Arriaga et al. (2010).

The approximate European option price which are described by system (10) can now be calculated using theorems 1-4.

Note that volatility $\sigma^{\bar{\varepsilon}, \bar{y}^{\prime}}$ of the option with price $w^{\bar{\eta}, \overline{\gamma^{\prime}}}(t, x ; K)$ is solved by the formula:
$w^{\bar{\eta}, \overline{\gamma^{\prime}}}(t, x ; K)=w^{B S}\left(t, x, \sigma^{\bar{\eta}, \overline{\gamma^{\prime}}} ; K\right)$,
where $w^{B S}\left(t, x, \sigma_{\bar{\eta}}^{\bar{\eta} \cdot \bar{y}^{\prime}} ; K\right)$ is Black-Scholes price with volatility $\sigma^{\bar{\eta}, \bar{\gamma}^{\prime}}$.

Note that figures are constructed component-wise in each corresponding time scale, similarly to both components in works of Lorig (2014), Burtnyak and Malytska (2016).

## CONCLUSION

This paper expands the methodology of approximate pricing for a wide range of derivative assets. Using a spectral theory of self-adjoint operators in Hilbert space and a theory of singular and regular disturbances, an analytic formula for the approximate asset prices is established, which is described by the CEV model with stochastic volatility dependent on $l$-fast variables and $r$-slowly variables, $l \geq 1$, $r \geq 1, l \in N, r \in N$ and a local variable. Applying the analysis of singular and regular perturbations at different time scales, we obtained explicit formulas for derivatives price approximations. The theorem of closeness estimates of financial instruments approximate prices is proved.

The main advantage of our pricing methodology is that by combining methods from spectral theory, regular perturbation theory, and the theory of singular perturbations, we reduce everything to the solution of the equations to find their Eigen functions and eigenvalues.

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